Fuzzy Hexagonal Automata and Snowflakes

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Abstract
There is a common perception that snowflakes are 6-sided and a common quip that "no two snowflakes are alike". The 6-fold symmetry suggests that the growth is deterministic, while the differences suggest the growth is so sensitive to conditions that remarkable variety appears. We investigate fuzzy automata that give a great diversity of growth patterns, have sensitivity to background conditions, and which maintain the symmetry of snowflakes.

Introduction

As early as 1611 Kepler wrote about the mystery of the six-symmetry of snowflakes, suggesting that it was related to the close packing of spheres [1]. Modern atomic theory makes it clear that the hexagonal packing does play a key role in ice crystal shape [2]; however, there remains much mystery in the details of the intricate designs we see in snowflakes.

Cellular automata have been viewed as models of growth for many years. Perhaps the most famous automaton, the Game of Life [3], yields rich and complex patterns that continue to generate ongoing study [4]. Mackay suggested in 1976 that automata could be used to simulate snowflake growth [5]. He illustrated with a simple branching tree structure created by Ulam using an automaton on a square lattice. He went on to suggest ideas for rules and that vapor pressure and convexity should play important roles. Wolfram has much more recently written at length on automata [6]. He suggests that simple automata are responsible for much of the complexity that we observe in nature. In particular, he describes the simple and impressive snowflake automaton in the style of Packard [7,8]. However, that automaton leads to limited diversity. They suggest that using larger neighborhoods or accounting for temperature rises near recently added ice might enhance the reality. We recently implemented this Packard-Wolfram snowflake automaton in J and generalized it to various Boolean automata on a hexagonal array [9]. While the behavior was in some cases wild, there did not seem to be diverse, complex behavior resembling growth in those Boolean automata. Here we consider some generalized automata that maintain six-fold symmetry and which exhibit diverse growth patterns.

The cells of the automata we will consider are arranged in a hexagonal lattice and each cell will be allowed to have a fuzzy value; that is, any value between 0 and 1. The automata use simple local arithmetic combinations, depending upon the neighborhood configuration, so that the six-fold symmetry persists. We are interested in exploring the diversity of growth patterns that these automata generate. In particular, we are interested in sensitivity to background conditions. While we have an eye towards the kind of diversity seen in real snowflake growth, we are not modeling the physics of snowflake

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growth. However, we will see that these automata may produce complex and varied growth patterns, some of which are quite suggestive of snowflakes.

Snowflakes

Natural snowflakes exhibit a wide range of behaviors. Classic examples appeared in Bentley and Humphreys [10] who photographed thousands of snowflakes over a period of decades. Several examples appear in Figure 1. We see several classic hexagonal plates and various stellar forms. In addition, there are imperfect forms with 12-fold, 4-fold, and 3-fold symmetry and a there is a hexagonal column with plates as end-caps. The hexagonal column makes the point that snowflakes may have non-trivial 3-dimensional features that presumably can not be captured by 2-dimensional models.

Nakaya studied and recorded snowflake growth under a wide range of conditions both in nature and in the laboratory [11]. His work initiated the modern classification of snowflakes. Snow crystals often are imperfect and the *Field Guide to Snow Crystals* gives a practical guide to natural snowflakes [12]. Real snow modifies as it ages and natural ice crystals need not be created via flakes falling through the air. For example, rime forms on the forward edges of obstacles to the wind under many winter conditions. Figure 2 shows some rime on Mt. Skylight in the Adirondacks. Temperature-gradient metamorphism of fallen snow can lead to depth hoar, which has a weak crystalline structure that contributes to avalanche formation.

The physics of snowflake growth is complex and not entirely understood. Ken Libbrecht's excellent resource on snowflake growth appears at [13]. The best 2-dimensional models appear to use global techniques (not automata) [14]. It seems that fully modeling 3-dimensional snowflake growth is beyond the current state-of-the-art. Thus, while snowflake growth and metamorphism are important, simple local models that exhibit the diversity of growth seen in nature seem not to have been implemented. We will see that our fuzzy models, like snowflakes, offer significant diversity in growth patterns.

A Boolean Model

Before turning to the fuzzy model that is the focus of our investigation, we will briefly discuss the Boolean Packard type snowflake automaton as described by Wolfram [6]. In that model, unfrozen cells remain unfrozen and frozen cells remain frozen except that an unfrozen cell with exactly one frozen neighbor becomes frozen. Figure 3 shows the evolution of a single frozen cell at iterations 63, 95, 109, 112, 118 and 127. An animation of the evolution may be seen at [9]. Notice that the first and the last are very similar except for scale. Also, each image is similar in density and the branching and gaps are limited in structure and size. Nonetheless, this simple Boolean automaton does have features suggestive of snowflakes, even if it is far from showing a wide range of diversity.

Fuzzy Automata with Snowflake Symmetry

We seek to find fuzzy automata that maintain the symmetry of the snowflake but also show wide diversity of behavior. Our first attempts used fuzzy automata on a hexagonal lattice where neighborhoods consisted of immediate neighbors. Even when we attempted to enhance those models to account for heating near locations recently frozen, we were unable to get the diversity of growth that we sought. We eventually settled upon using two levels of neighbors with a local summing technique. Since the values of our cells may be any value between 0 and 1, we take 0.5 as the threshold for being frozen (the fuzzy values may be perceived as a fuzzy estimate for whether the location is part of the snowflake). Our standard experiment involves setting the entire background to some fuzzy value between 0 and 0.5 and setting a single cell to 1. We then let the automata evolve and create suitable animations of the evolution.

For purposes of describing our algorithm, we need the following distinctions. Each neighborhood is divided into four types of entries. A sample neighborhood is shown in Figure 4. There is a single central cell (marked *C*) and there are 6 immediate neighbors (marked *N*); there are 12 neighbors at the next level and those are subdivided into two types: the six corners or "points" (marked *P*) and the six edge entries (marked *E*).

In deciding on how the automata evolves, we consider several things. Let *C* denote the Boolean value (0 for no, 1 for yes) which specifies whether the center is frozen; let *N* denote the number of frozen *N*-neighbors; and let *P* denote the number of frozen *P*-neighbors whose adjacent *N*-neighbor is also frozen. We select different weights depending upon the values of *C*, *N* and *P*. If $(\alpha, \beta, \gamma, \delta)$ give the weights for some choice of *C*, *N* and *P*, then compute the linear combination that is given by $S = \alpha SC + \beta SN + \gamma SP + \delta SE$ where SC is the fuzzy value of the center, SN is the sum of the six fuzzy values at the *N*-neighbors, and likewise for SP and SE. If S is between 0 and 1, then S is the new value of the cell; if $S < 0$, the new value is 0; if $S > 1$, then the new value is 1.

Notice that the labeling of neighbor types in Figure 4 has symmetry. In particular, it has the dihedral, D_6, symmetry that we associate with snowflakes. That is, there is 6-fold rotational symmetry around the center and reflection symmetry through any pair of opposite vertices. Since the weights for the linear combination must be the same for each entry of the same neighborhood entry types, this D_6 symmetry will be preserved at each step, given the input pattern also has that symmetry.

In practice, there are a large number of weights that we may specify. The eleven different (*C*, *N*, *P*) cases that we considered are shown in the left column of Table 1. Each case has its $(\alpha, \beta, \gamma, \delta)$ weights; although in practice, the weights that we use are often the same for several of the eleven cases. In practice, we take the (*C*=0, *N*=0, *P*=0) case to be weighted so that the center value remains unchanged. That is, $(\alpha, \beta, \gamma, \delta) = (1,0,0,0)$ for that case. This allows the fuzzy initial values away from the initial frozen cell to remain unchanged until a frozen cell is nearby. Thus, in our implementation we were free to select the parameters in the other ten cases.

It is easy to select the $(\alpha, \beta, \gamma, \delta)$ to duplicate the Boolean automaton described in the previous section. For example, one could take $(\alpha, \beta, \gamma, \delta) = (1,0,0,0)$ for the cases when *N* is not 1, and $(\alpha, \beta, \gamma, \delta) = (0,1,0,0)$ when *N*=1. Similar choices can be made to give that automaton for some fuzzy backgrounds, but not others.

A Detailed Example

We consider a particular choice of parameters that nicely illustrates the sensitivity to background level. Table 1 shows the choice of the particular parameters. While the parameters were determined experimentally, we can make a few remarks upon the choices. Notice that the parameter corresponding to ice with no neighbors leads to slow increase in the fuzzy value. Also notice the favoritism given to growth when P=1.

(C, N, P) range	($\alpha, \beta, \gamma, \delta$)
(C=1, 0≤N≤6, 0≤P≤N)	(1, 1/300, 0, 0)
(C=0, N=0, P=0)	(1, 0, 0, 0)
(C=0, N=1, P=0)	(1, 1/6, -1/12, -1/12)
(C=0, N=1, P=1)	(1/5, 3/10, 1/10, -1/10)
(C=0, N=2, P=0)	(1/20, 2/5, -3/20, -3/20)
(C=0, N=2, P=1)	(1/10, 3/10, -1/10, -1/10)
(C=0, N=2, P=2)	(1/22, 4/11, -3/22, -3/22)
(C=0, N≥3, P=0)	(1/18, 4/9, -1/6, -1/6)
(C=0, N≥3, P=1)	(1/10, 3/10, -1/10, -1/10)
(C=0, N≥3, P=2)	(1/22, 4/22, -3/22, -3/22)
(C=0, N≥3, P≥3)	(1/18, 4/9, -1/6, -1/6)

Table 1. Parameter choices for the example fuzzy automaton shown in Figures 5-7.

Figures 5 and 6 show the growth of the automata for several background values. The color shown at a cell primarily corresponds to the fuzzy value of the cell at iteration 127; however, we also show some color change corresponding to thick ice (fuzzy value 1), which shows how long it has been thick. Colors from darkened cyan through green shades correspond to fuzzy values below 0.5. Since there is little variation in these values as the automaton evolves, typically no variation in those levels is visible. Fuzzy values from 0.5 up to 1 are shown with hues running from yellow to red. Shades from red to purple correspond to thick ice with the color corresponding to the length of time the flake has had maximal value of 1. Red values just became thick, while the most purple ones have been thick for 127 steps.

Figure 5 shows the automata described by the parameters when a single cell at value 1 is imbedded in a fuzzy background with values: 0.156, 0.159, 0.163, 0.165, 0.17, 0.173, 0.1758, 0.1762, and 0.1771. Notice the great diversity of forms that appear even though the parameter values are quite close. Some of these are quite suggestive of the forms seen in snowflakes. In general, but not always, there is slow growth for small background levels and fast growth for high backgrounds. The growth for levels below 0.15 is significantly slower than what we see in Figure 5. Figure 6 shows the growth of the automata for backgrounds 0.1778, 0.1785, 0.1788, 0.18, 0.184, 0.188, 0.2, 0.24, and 0.3. Since the illustrations in Figure 5 grow more slowly, they are imbedded in a smaller background. Thus, images in Figure 5 should be viewed as being magnified 1.5x when compared to the images in Figure 6. By the time that the background level approaches 0.3, we see the forms are more plate-like. Animations of the growth of these forms may be found at [15].

Choosing different parameter values leads to many other illustrations, but we chose the parameters in Table 1 since there is interesting behavior over a wide range of background levels. Even without changing the parameters in the growth model, we will see in the next section that there are many ways to produce many further variations.

Variants and Implementation

Writers on snowflake growth have suggested that the variety seen among snowflakes is due to the subtle differences in conditions as the flake falls. We can simulate this by varying the fuzzy background level between different values. Figure 7 shows illustrations of 6 cases where, in the background, we have oscillated between two fuzzy values every 16 steps away from the seed ice point. Technically this is a spatial oscillation, not a temporal one, but the ideas are very similar. In Figure 7 and additional experiments we see that our example automaton is consistent with that expectation: it gives a wide variety of additional behaviors on such backgrounds. The pairs of background values for those illustrations were 0.1758 &0.3000, 0.1771&0.2000, 0.1840&0.2000, 0.2000&0.1800, 0.2000&0.3000, and 0.2400&0.1630.

We noted earlier that real snowflakes are not necessarily perfect and we saw in Figure 1 examples of snowflakes with apparent 3-fold, 4-fold and 12-fold symmetry. While we can not embed 12-fold symmetry into our hexagonal lattice, we can easily set 2, 3 or 4 neighboring points to have initial value of 1 and thereby slightly break the symmetry of the initial configuration (giving 2-fold, 3-fold and 2-fold symmetry, respectively). Figure 8 shows examples of doing this where the background levels are 0.173, 0.163 and 0.165, respectively. Notice the pairs and quadruples of features in the first and third images. The middle example has 3-fold symmetry preserved.

We implemented our experiments using Jsoftware and created the animations using the Image3 addon [16]. A Jsoftware script that may be used to create an image from Figure 6 (middle row, rightmost) may be found at [15].

Conclusions

We have seen that we can implement fuzzy automata on a hexagonal background using simple arithmetic combinations of nearby fuzzy values. Our main example shows great sensitivity to background level. Thus we see a wide variety of behaviors, some quite similar to snowflakes, as the background is varied. Using non-constant background patterns yields still further variety. Moreover, using less symmetric initial frozen patterns give similar growth with somewhat less symmetry. Thus we see it is possible to create simple fuzzy automata that maintain the symmetry of snowflakes and exhibit great diversity depending upon the background in which they grow.

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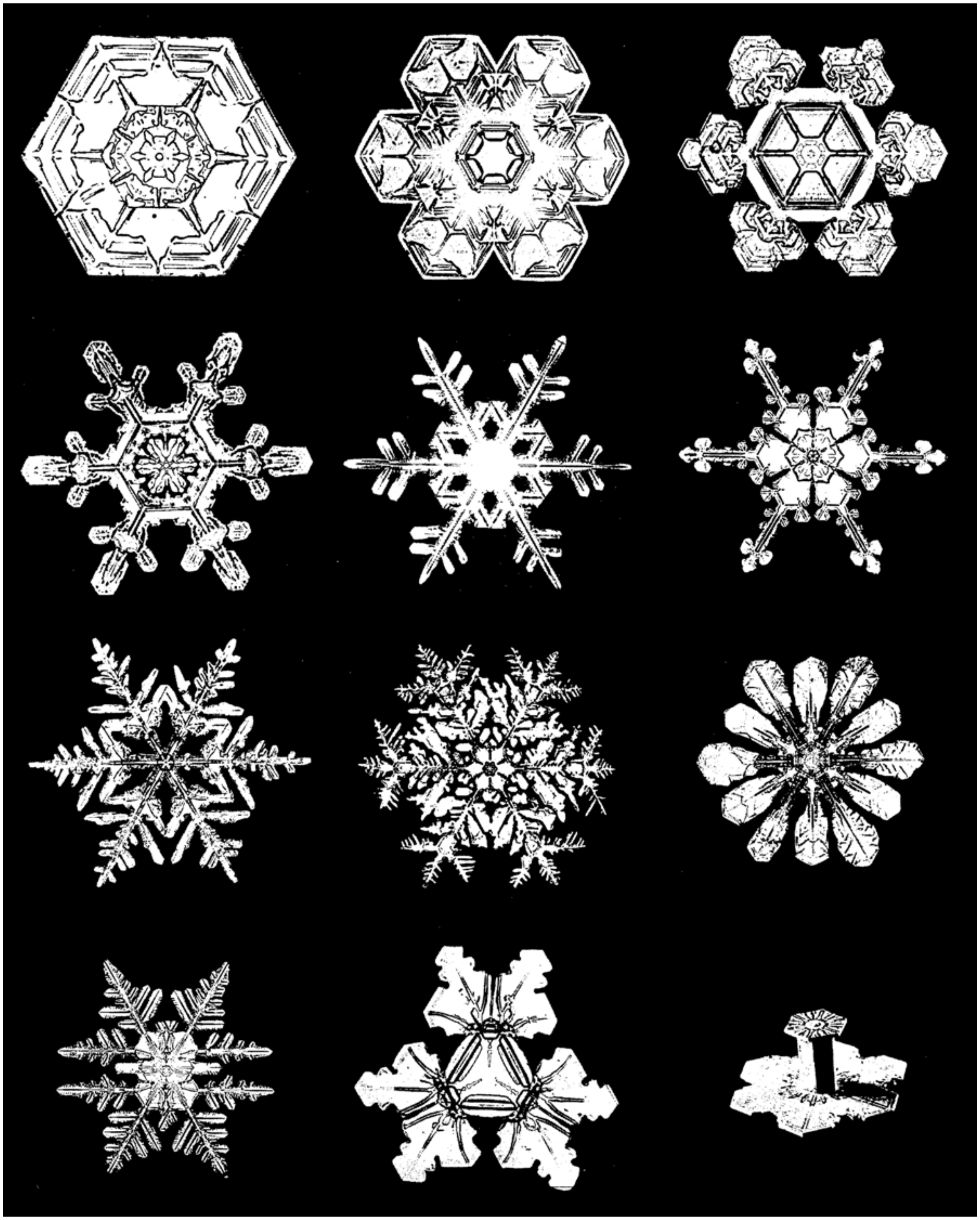

Figure 1. Some snow crystals

Figure 2. Some Rime on Mt. Skylight

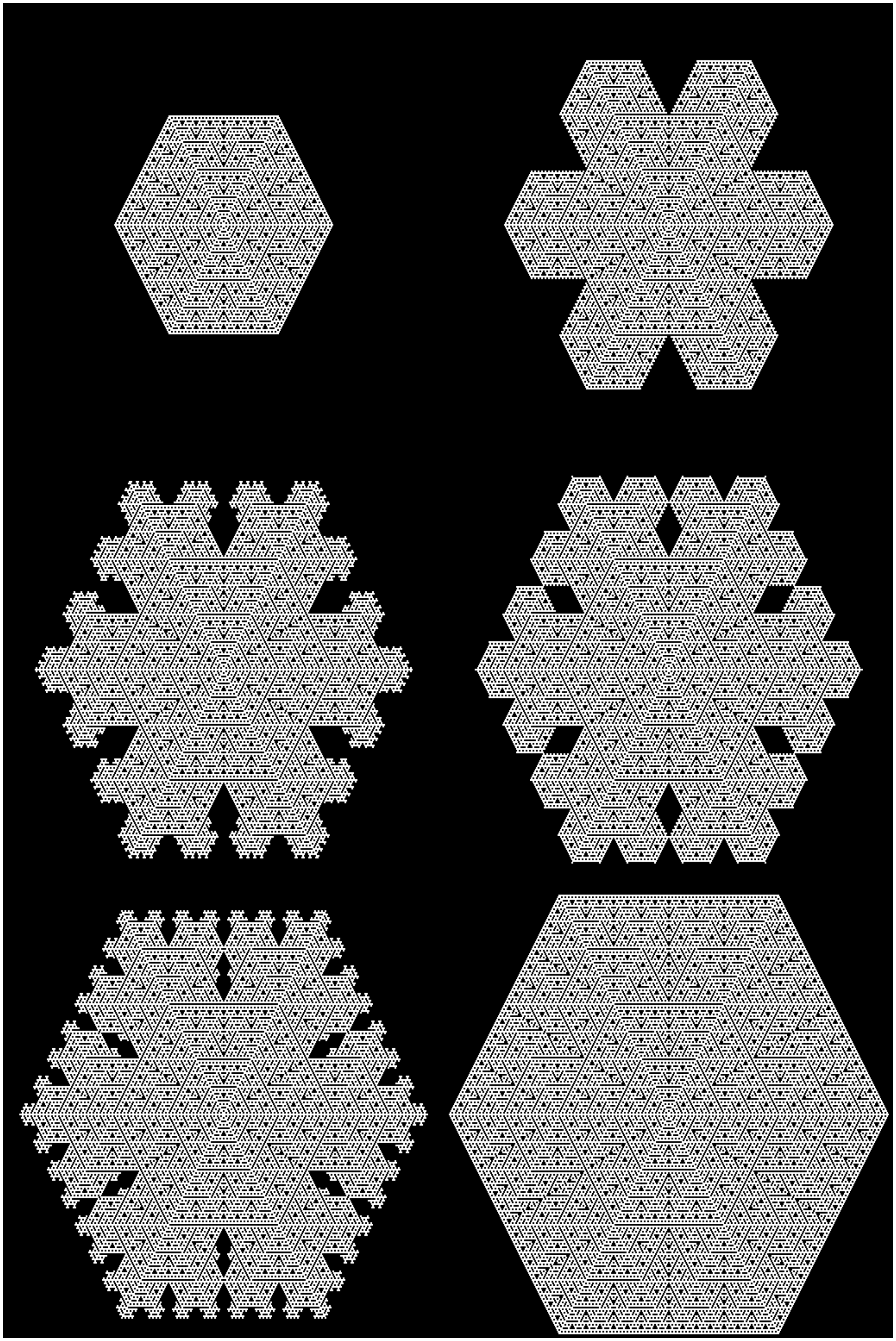

Figure 3. Packard-Wolfram Boolean snowflake automata at iterations 63, 95, 109, 112, 118 and 127.

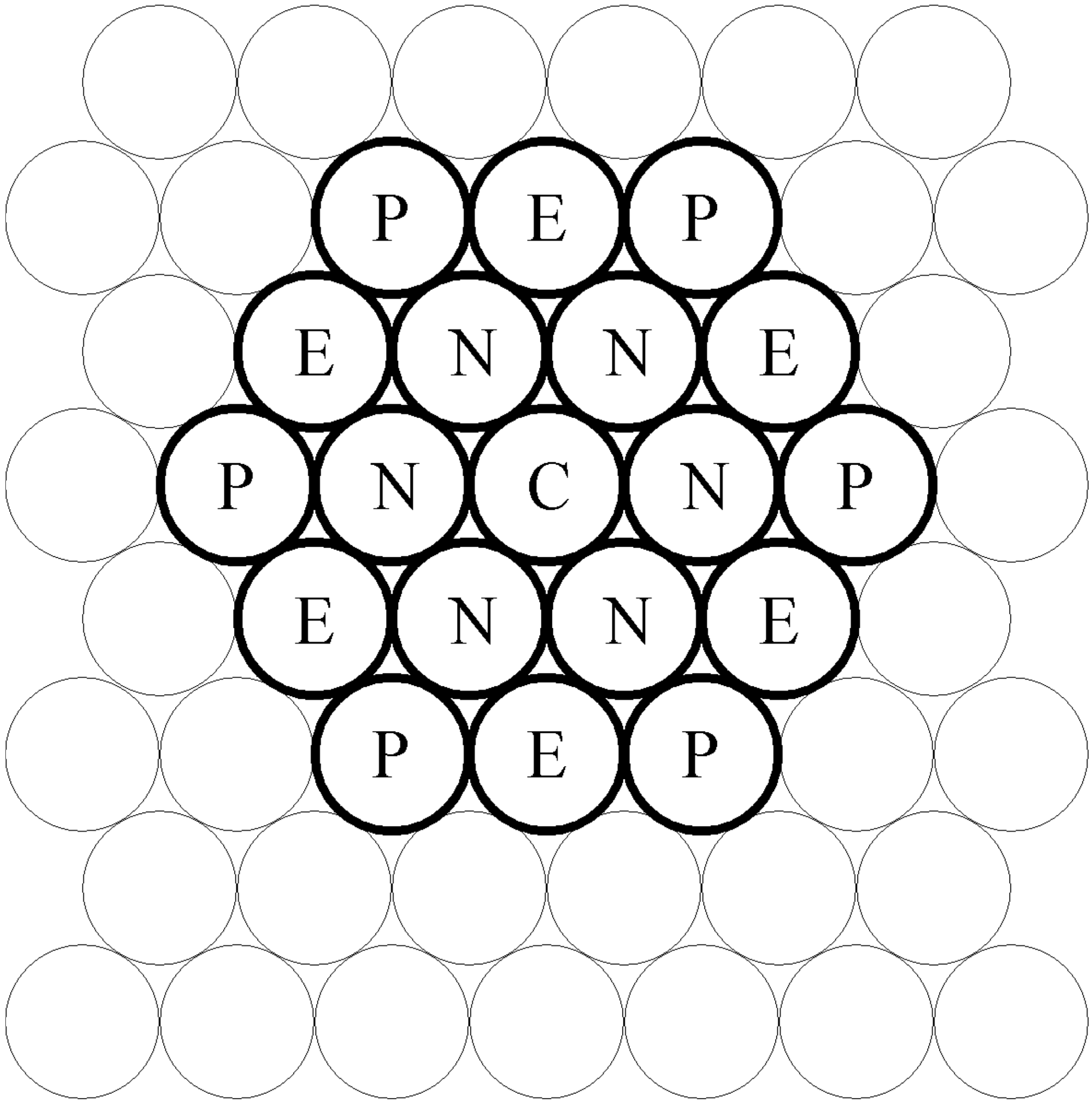

Figure 4. The entry types of a local neighborhood, sitting in a hexagonal arrangement of cells.

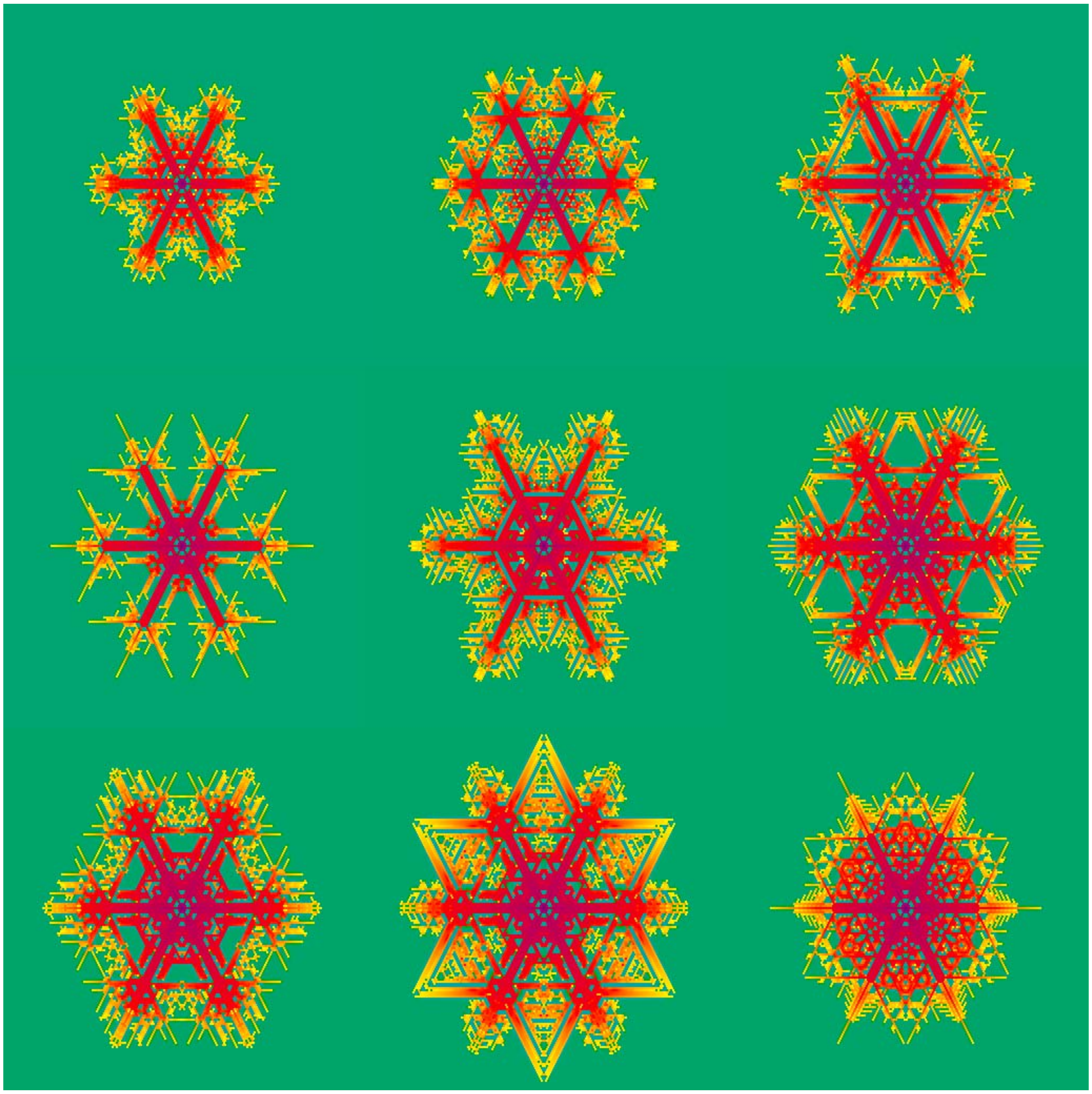

Figure 5. Evolution of a fuzzy automata for background values 0.156, 0.159, 0.163, 0.165, 0.17, 0.173, 0.1758, 0.1762 and 0.1771. Shown at iteration 127 and with 1.5x magnification.

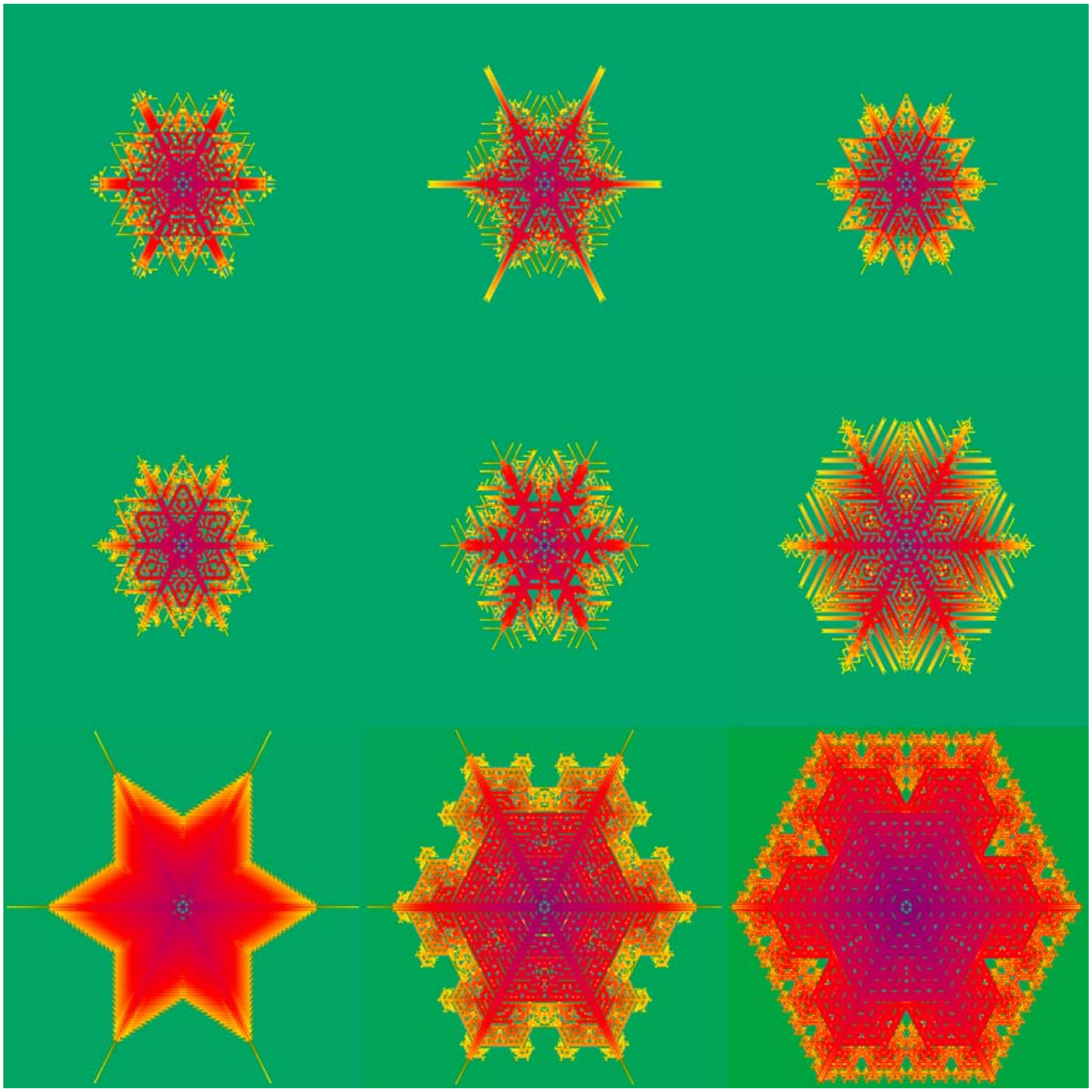

Figure 6. Evolution of a fuzzy automata for background values 0.1778, 0.1785, 0.1788, 0.18, 0.184, 0.188, 0.2, 0.24, and 0.3. Shown at iteration 127 and with no magnification.

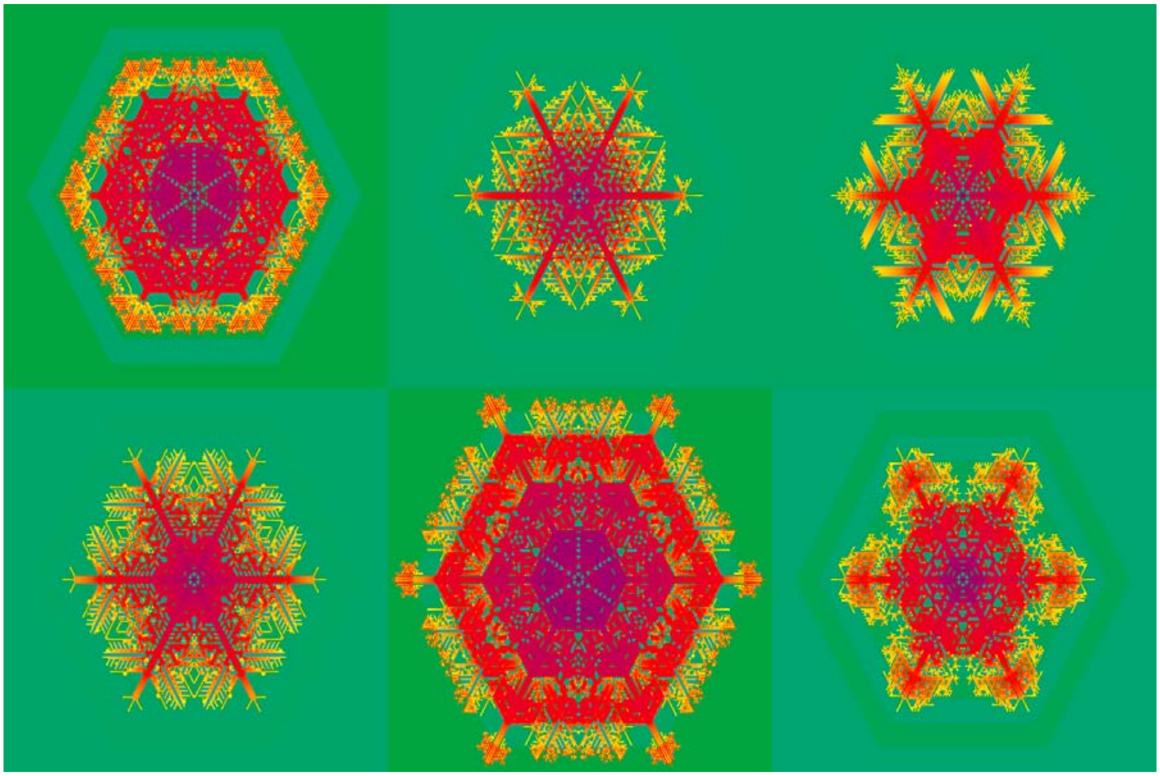

Figure 7. The growth on some backgrounds that oscillate between two fuzzy values.

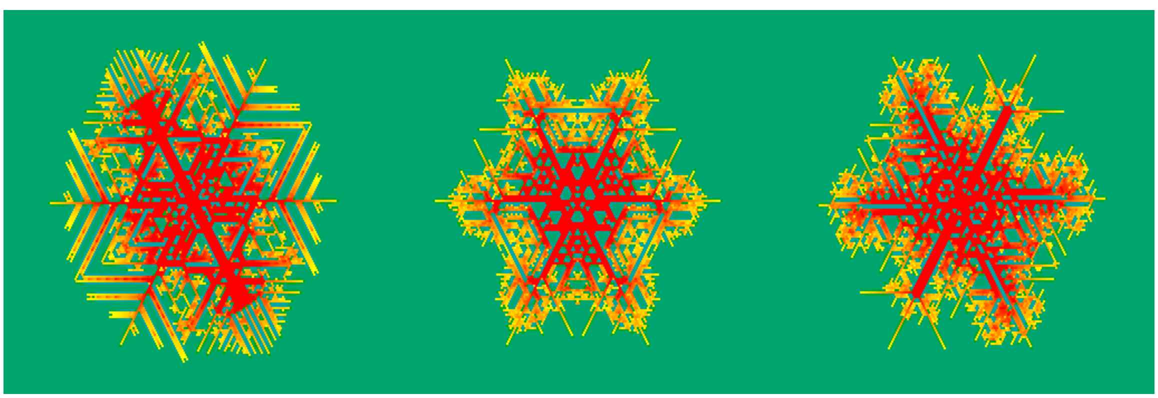

Figure 8. Growth on two, three and four seed points.