Cyclic Cellular Automata in 3D

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Abstract

Cyclic cellular automata in two dimensions have long been intriguing because they self organize into spirals and that behavior can be analyzed. The form for the patterns that develop is highly dependent upon the form of the neighborhood. We extend this work to three dimensional cyclic cellular automata and observe self organization dependent upon the neighborhood type. This includes neighborhood types intermediate between Von Neumann and Moore neighborhoods. We also observe the patterns include nested shells with the appropriate forms but that the nesting is far more complex than the spirals that occur in two dimensions.

Keywords: Cyclic Automata, Cyclic State, Self-organizing, 3D automata
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1. Introduction

A cellular automaton is a collection of cells each of which has neighbors. At every time step each cell is in a state and the set of allowed states is usually finite. At each time step the states of all cells are updated according to a local rule that depends only upon the state of the cell and its neighbors. Cellular automata are valuable because they may be described using simple local rules yet the global behaviors that evolve from those rules can be extremely rich [1, 2, 3]. Perhaps the most famous cellular automata is Conway's Game of life that was popularized by Martin Gardner's Scientific American columns [4, 5] and has since been shown to be capable of universal computation [6, 7, 8].

Dewdney described cyclic cellular automata [9] in Scientific American in 1989 where they were called cyclic state automata. He states they were discovered by David Griffeath who subsequently investigated them with others [10]. We will refer to them as cyclic cellular automata (CCA). When CCA are applied to a random initial configuration in 2D, they typically evolve through distinct phases that have different appearances. The end result is visually dramatic, periodic spirals that self-organize. In [11] those automata were investigated using a wide variety of neighborhoods including asymmetric and quasicrystaline neighborhoods. Generally speaking, the general form of the spirals echoed the neighborhood shapes in the sense that their form roughly followed the form of an inverted outline of the neighborhood, but spiraling and periodic.

The fact that cyclic cellular automata on random backgrounds are expected to evolve into a periodic organized state can be explained [10, 9, 11] as will be tersely described in the next section. Readers interested in more complete explanation should consult those references. The same analysis means that CCA on 3-dimensional rectangular grids should also evolve through phases to a final nontrivial periodic configuration. Previous theoretic work on 3-dimensional periodic spirals motivated by chemical reactions shows that there are limits on possible configurations and an exclusion principle is developed [12, 13]. In this paper we experimentally explore the 3-dimensional analogs of spirals and see that they depend upon neighborhood shape. We also see that demons have a great deal more freedom to evolve in complex forms.

2. Cyclic Cellular Automata and Terminology

A cyclic cellular automaton is defined as an automaton where each cell takes one of N states $0, 1, 2, ...N-1$ and a cell in state i changes to state $i+1$ *mod* N at the next time step if it has a neighbor that is in state $i+1$ *mod* N, otherwise it remains in state i at the next time step.

The bond between two neighbors is *open* if the difference between their states is $-1, 0, 1$ *mod* N. Otherwise the bond is *closed*. Note that once a bond is open it must remain open for all future time. A site is considered *debris* at a given time if it has no open bonds with its neighbors. The connected components of the non-debris sites are called *droplets*. A loop within a droplet using open bonds could possibly include only bonds $-1, 0, 1$ *mod* N. If one takes a running sum (not *mod* N) of the differences *mod* N (so each difference is from $-1, 0, 1$) along such a loop, then the total must be zero *mod* N since each cell value appears once in a positive and negative sense. Thus, the sum of the differences must be a multiple of N. If the multiple is not zero, then the loop is part of a *defect*. That property of the loop will be preserved as the automaton evolves, and each cell in a defect must eventually continue cycling through the states forevermore; although it may take more than one time step between each change of state. Eventually every cell neighboring such a loop must cycle. Then their neighbors must cycle. Eventually every cell in such a droplet must cycle, and thus the droplet must grow since every neighbor of the droplet will eventually join the droplet since the neighbor in the droplet will eventually cycle through all values. Of course the speed of such cycling need not be constant. If a defect loop runs through all states as efficiently as possible on the lattice (minimal period having no unnecessary 0-difference bonds) then it is a *demon*. Given random initial states for all the cells on an infinite lattice, demons are expected to occur somewhere with probability one. Thus, demon domination is the expected long term state.

We illustrate these notions with a 2-dimensional example. Figure 1, from [11], shows the evolution of a $N = 14$ state CCA with Von Neumann neighbors and periodic boundary conditions on a 500 by 500 array of cells. The states are shown cyclically with hue, running from red through green and blue and magenta. In the upper left time step 75 is shown and debris and droplets can be noted. In the upper right time step 150 is shown and some debris remains, droplets dominate, but a few spirals have self organized. In the lower left time step 225 is shown and the spirals have grown and several new spirals have formed. Portions of the image that have reached periodicity 14 are separated from other portions by the black pixels. In the lower right time step 975 is shown and spirals dominate and the optimal period 14 spirals have almost overtaken all the higher period spirals. The spirals have a diamond orientation, which is the outline of a cell's Von Neumann neighbors.

3. Cyclic Cellular Automata in 3-Dimensions

In a rectangular three dimensional grid there are various highly symmetric choices for neighborhoods. Imagine a 3 by 3 by 3 block of cells numbered as in the three planes shown below.

0 1 2	9 10 11	18 19 20
3 4 5	12 13 14	21 22 23
6 7 8	15 16 17	24 25 26

We consider the center cell to be in position 13 and the six Von Neumann neighbors share "faces" and have positions 4, 10, 12, 14, 16 and 22. The

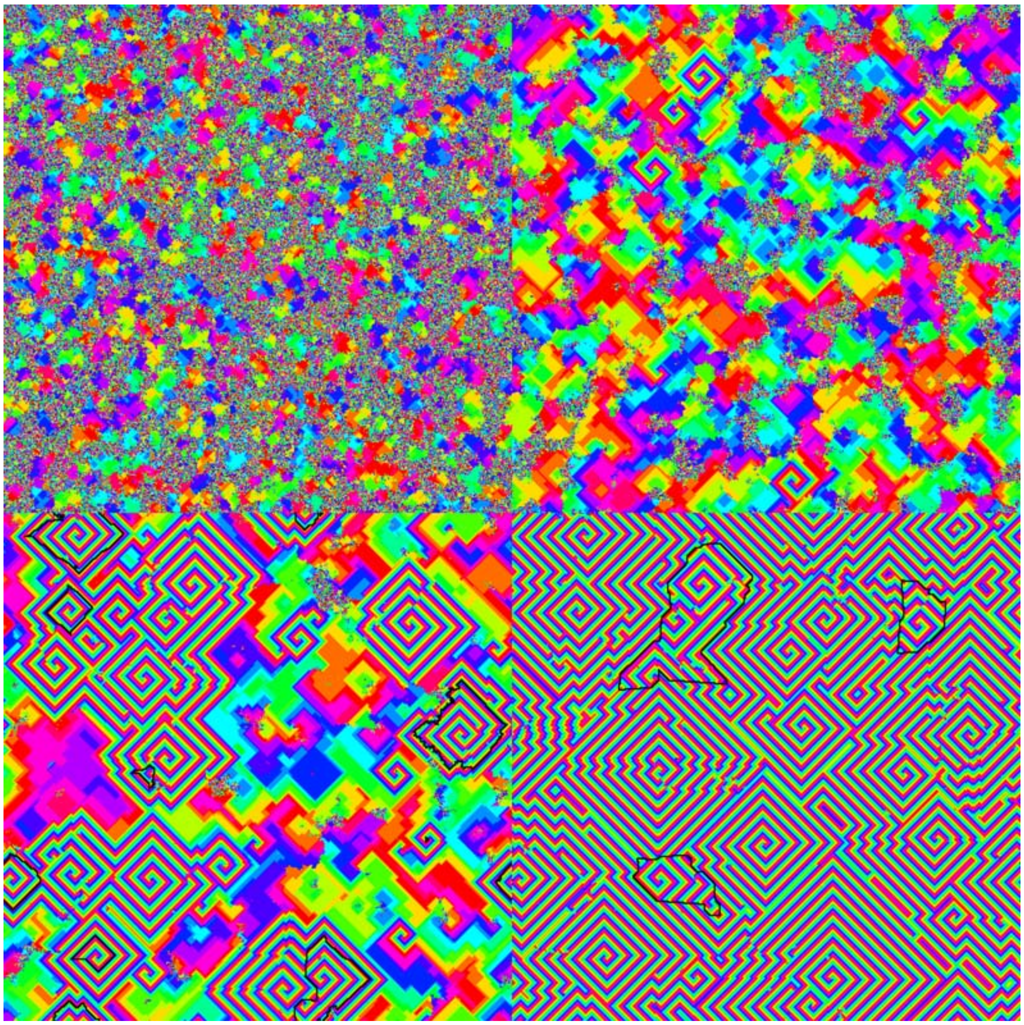

Figure 1: Von Neumann CCA in two dimensions with $N = 14$ states at times $t = 75, 150, 225, 975$

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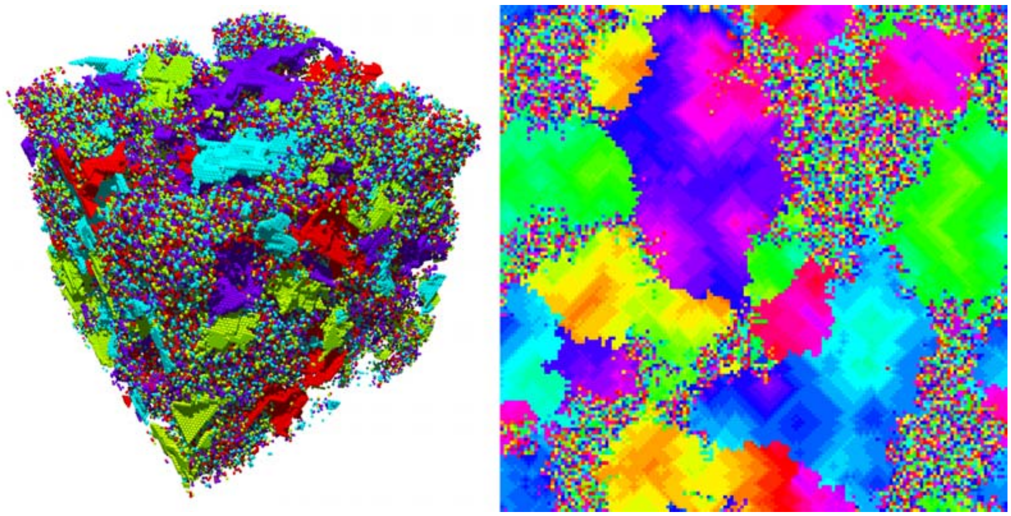

Figure 2: Von Neumann CCA in three dimensions with $N = 38$ states at time $t = 400$
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Moore neighbors are all the positions except 13. In 3-dimensions we also have the intermediate "edge" neighbors which amount all the positions except the eight cube corners. That is, 0, 2, 6, 8, 18, 20, 24, and 26 are not included.

We first consider the Von Neumann neighbors. The outline of those neighbors has an octahedral shape. Figure 2 shows the configuration using $N = 38$ states after $t = 400$ steps are applied to an initially random 150 by 150 by 150 configuration. On the left is a 3-dimensional view with just 4 states shown while on the right the entire top plane is shown. Debris and development of droplets is apparent. Figure 3 shows the configuration after $t = 500$ steps where debris, droplets and defects are all visible. Figure 4 shows $t = 999$ where nested octahedral-like faces have developed. Notice a kind of spiraling of the level can be observed. Movies showing the full evolution of this automaton may be found at [14].

Figure 5 shows the closed bonds that arise from the configuration shown in the previous three figures after period 38 has been achieved. Any circuit in the transparent region that circuits a column would be a demon. The almost fractal like pattern the closed bonds exhibit suggests there is a rich variety of demons.

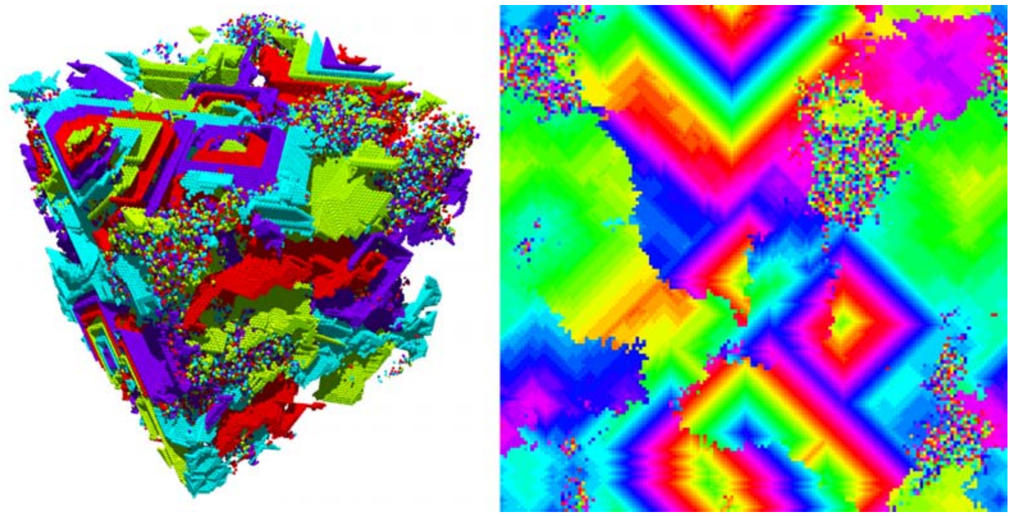

Figure 3: Von Neumann CCA in three dimensions with $N = 38$ states at time $t = 500$

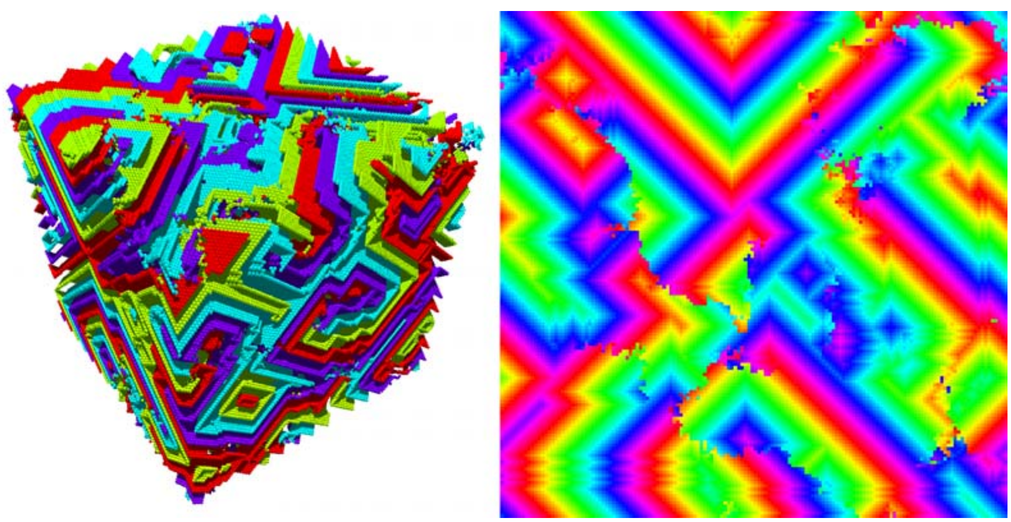

Figure 4: Von Neumann CCA in three dimensions with $N = 38$ states at time $t = 999$

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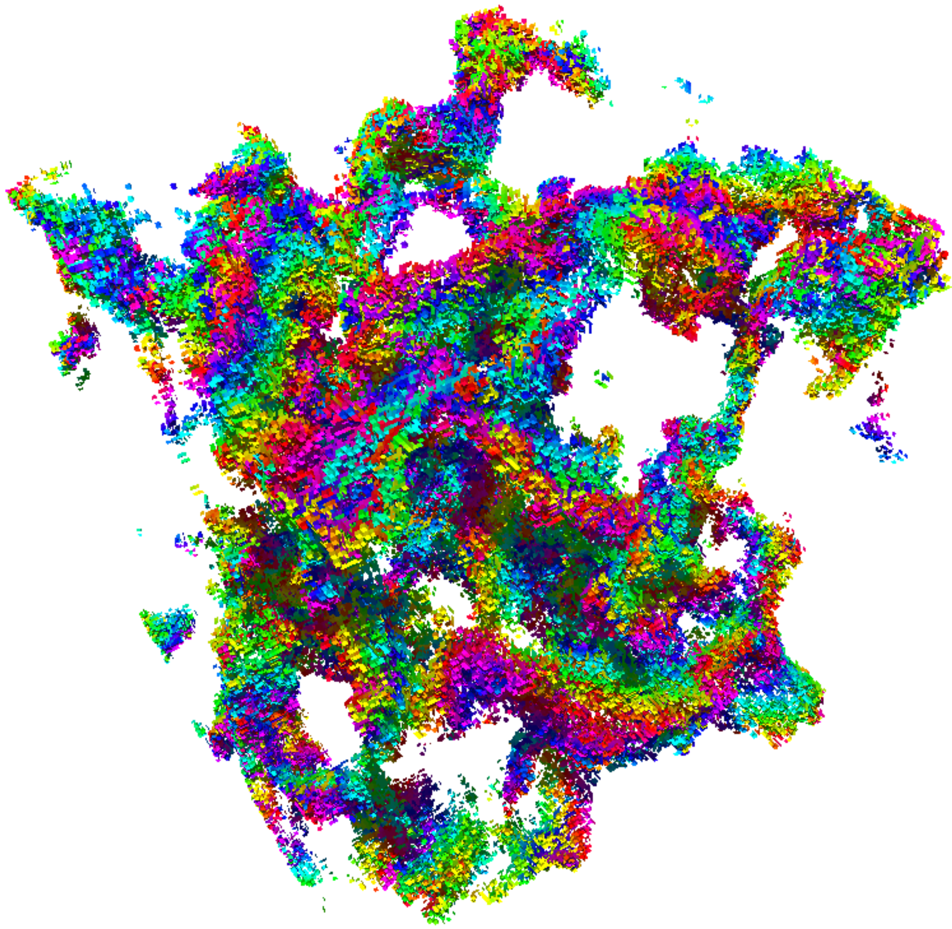

Figure 5: Closed bonds for Von Neumann CCA in three dimensions with $N = 38$

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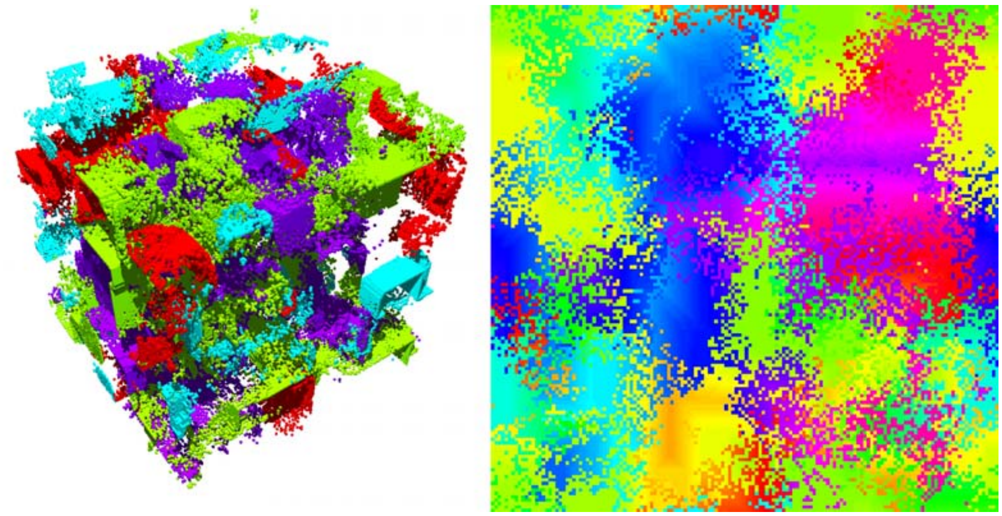

Figure 6: Moore CCA in three dimensions with $N = 128$ states at time $t = 642$

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We next consider the Moore neighborhoods using $N = 128$ states. The outline of those neighbors has a cubic shape. Figure 6 shows the configuration after $t = 642$ steps and complex passing waves with partial cubic-like forms observed in many places.

We next consider the Edge neighborhoods and $N = 86$ states. The outline of those neighbors has a truncated cubic shape with 14 faces. Figure 7 shows the configuration after $t = 534$ steps. Partial truncated cubic patterns appear and spiraling may be observed.

We lastly consider mixed neighborhoods that are Moore in two dimensions but Von Neumann in the orthogonal direction. In particular, the neighbors have positions 4, 9, 10 11, 12, 14 15, 16, 17, and 22. The outline of those neighbors has an unclear shape although we might expect square cross sections both parallel to and at a 45 degree offset from the underlying rectangular lattice. Figure 8 shows the configuration when there are $N = 56$ states after $t = 640$ time steps.

4. Conclusions

We have seen that Cyclic Cellular Automata in three dimensions must self-organize for the same reasons that 2-dimensional versions self-organize.

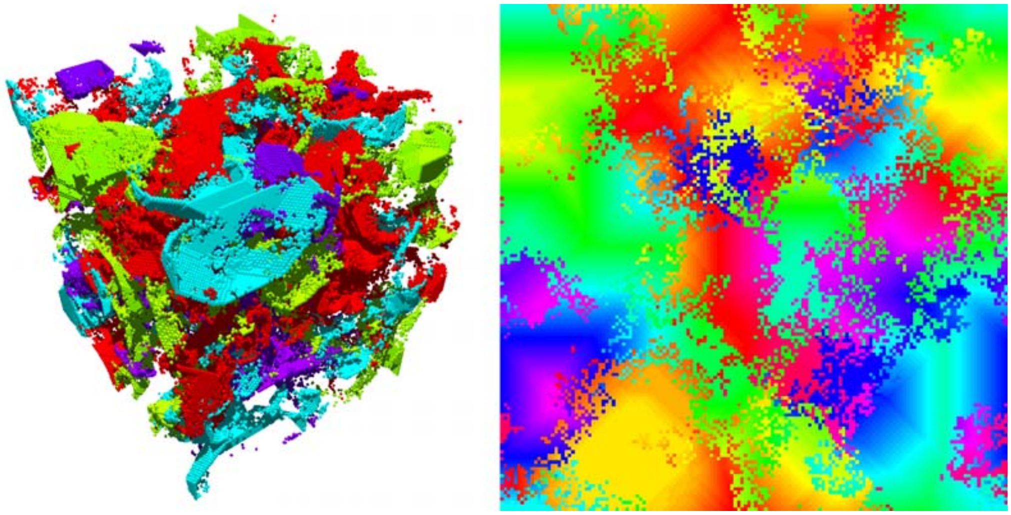

Figure 7: Edge CCA in three dimensions with $N = 86$ states at time $t = 534$

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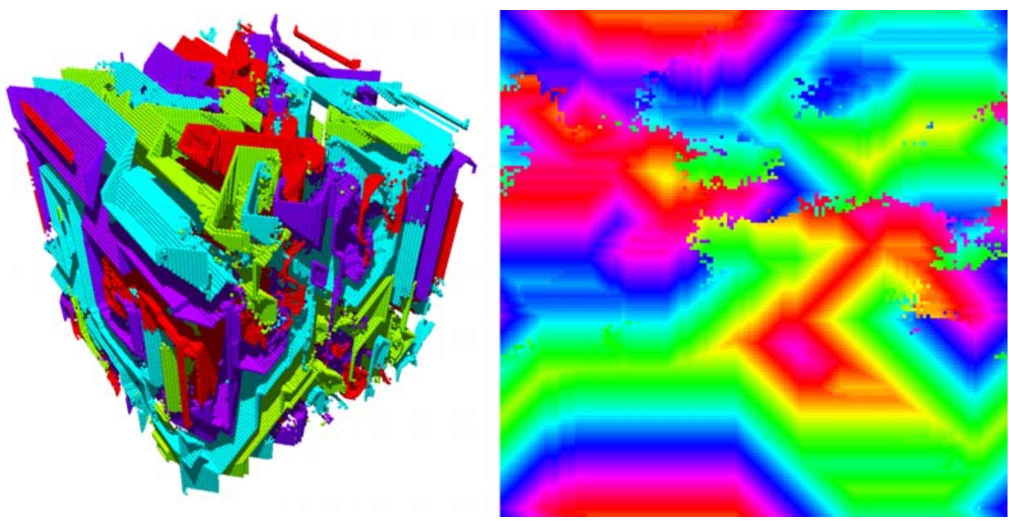

Figure 8: Von Neumann & Moore mixed CCA in three dimensions with $N = 56$ states at time $t = 640$

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We see that the shapes of the organization reflect the patterns of the neighborhoods and in three dimensions there are additional highly symmetric neighborhoods that are possible. However, in three dimensions the self organization is more complex than simple spirals since concentric levels of a demon must be connected, but in three dimensions the paths for those connections have many alternatives.

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