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REAL DYNAMICS OF A 3-POWER EXTENSION OF THE $3x + 1$ FUNCTION

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Abstract. This paper investigates the real dynamics of a generalization of the $3x + 1$ function using powers of three. We will see that any cycle of positive integers is attractive for this generalization and the cycle has an expansion factor given by Terras' coefficient function. We will see the function has a negative Schwarzian derivative for $x \geq 0$ and will be able to identify invariant intervals and approximately locate the fixed points and critical points. The special simplicity of dynamics around the cycle $(1, 2)$ means there is a natural generalization of total stopping time for this function. We conjecture that the odd critical points of this generalization are well behaved. In particular, they lie in the immediate basin of total stopping time surrounding each odd integer.

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AMS (MOS) subject classification: 26A18, 11B83, 37C25

1 Introduction

The classic $3x + 1$ problem concerns the iteration of a function that results in $3x + 1$ for odd x and $x/2$ for even x. The classic $3x + 1$ conjecture is that iteration of that function upon a positive integer eventually reaches the cycle containing 1. However, there are slight variations on the function and there are many related conjectures now in the literature. Lagarias gives an overview of important early results regarding the $3x + 1$ problem in [5] and maintains an annotated bibliography of the subject [7]. The literature of the subject is rapidly growing and includes Wirsching's book [14], which is another source that provides an overview of the literature. Considerable amounts of information on the $3x + 1$ problem appear on the web; good starting points include the web version of [5] and Roosendall's site which includes search results [11].

Notice that when x is odd, then $3x+1$ is even, and hence the next iteration of the classic function will be division by 2. Thus, many of the results known about the $3x+1$ problem may be easily described in terms of the following function, which we will call "the" $3x+1$ function.

$$t(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \bmod 2 \\ \frac{3x+1}{2} & \text{if } x \equiv 1 \bmod 2 \end{cases} \quad \text{for } x \in \mathbf{Z} \tag{1}$$

In this form, the $3x+1$ conjecture becomes the conjecture that iteration of $t(x)$ on any positive integer eventually results in a value 1. Thus, primary interest in $t(x)$ is when $x > 0$.

The *total stopping time* of a positive integer, x, under iteration of $t(x)$ is the least nonnegative k, if it exists, such that $t^k(x) = 1$; otherwise, it is defined to be ∞. Thus, the $3x+1$ conjecture is equivalent to saying that every positive integer has finite total stopping time.

As noted in [13], the $3x+1$ function may be written in the form

$$\frac{1}{2}\left(3^{\bmod_2(x)}x + \bmod_2(x)\right) \tag{2}$$

where $\bmod_2(x)$ is 0 for even integers and 1 for odd integers. Notice that we can view Equation 2 as defining a function of a real or complex variable by taking $\bmod_2(x)$ as a real or complex function that is 0 for even integers and 1 for odd integers. In particular, in this note we investigate the real dynamics of the function $T(x)$ defined as follows.

$$T(x) = \frac{1}{2}\left(3^{\bmod_2(x)}x + \bmod_2(x)\right) \text{ where } \bmod_2(x) = \sin^2\left(\frac{\pi x}{2}\right) \tag{3}$$

We call $T(x)$ the 3-power extension of the $3x+1$ function.

Figure 1 shows the graph of $T(x)$. Notice that away from the origin there are regular growing oscillations between $x/2$ and $(3x+1)/2$ with fixed points in between. The function $T(x)$ was used in Terras [13] to derive the "remainder representation theorem" for the iteration of the $3x+1$ function. We will state the theorem in Section 2.

The dynamics of other functions interpolating the function $t(x)$ have been studied. Indeed, we consider Chamberlain's [1] interpolating function $C(x)$, defined in equation (4) below, to be the most direct interpolating function.

$$C(x) = (1 - \bmod_2(x))\frac{x}{2} + \bmod_2(x)\frac{3x+1}{2} \tag{4}$$

One can check that

$$C(x) - T(x) = \frac{x}{2}\left(2\bmod_2(x) + 1 - 3^{\bmod_2(x)}\right)$$

where the right-hand factor is nonnegative and periodic; from that it is easy to check that $\liminf_{x\to\infty} C(x) - T(x) = 0$ and $\limsup_{x\to\infty} C(x) - T(x) = \infty$. Thus, even though $C(x)$ and $T(x)$ agree infinitely often, namely, at the integers, they become unboundedly different in between.

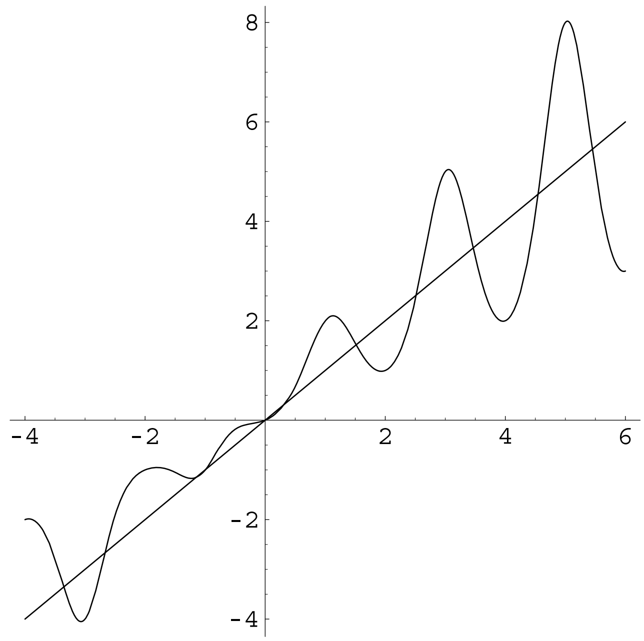

Figure 1: The function $T(x)$.

Chamberlain studied the real dynamics of $C(x)$ in [1]. The 3-power extension of the $3x+1$ function, $T(x)$, defined in Equation (3) is arithmetically more complicated than the function $C(x)$ because it involves exponentials; nonetheless, we will be able to establish facts about its dynamics analogous to those in [1]. However, we will also see that there are ways in which the behavior of $T(x)$ seems more natural.

We will show that any integer cycle of the $3x+1$ function is an attracting cycle of $T(x)$ and the multiplier for the cycle is the coefficient function of Terras. We will show that the Schwarzian derivative of $T(x)$ is negative for $x \geq 0$ and we can approximately locate the positive critical and fixed points. Unlike $C(x)$, the function $T(x)$ appears only to have a single attractive positive real cycle. Namely, the expected cycle $(1, 2)$. We also offer evidence for our conjecture that the odd critical points "shadow" the odd integers.

Before turning to our investigation of $T(x)$, we comment that some investigations of the complex dynamics of similar generalizations of the $3x+1$ function have appeared. Letherman, Schleicher and Wood [8] analyze a family of functions that generalize $t(x)$ that have complex dynamics that can be somewhat controlled. However, their family does not include either $C(x)$ or $T(x)$. In [4] the complex dynamics of several functions related to and including $C(x)$ and $T(x)$ were visually investigated.

2 Attractive Cycles

In this section we establish that any cycle of positive integers for the $3x+1$ function is an attractive cycle for $T(x)$. We will see that the expansion factor for a cycle of integers is the coefficient function from Terras and it must be less than one.

Theorem 1 *Any positive integer cycle of the* $3x+1$ *function is an attractive cycle of* $T(x)$.

Proof. Note that if Ω is a cycle of positive integers for the $3x+1$ function, then

$$1 = \prod_{x\in\Omega} \frac{T(x)}{x} = \prod_{x\in\Omega} \left(\frac{1}{2}3^{\text{ mod }_2(x)} + \frac{\text{mod}_2(x)}{2x}\right) > \prod_{x\in\Omega} \frac{1}{2}3^{\text{ mod }_2(x)} \tag{5}$$

since $\frac{\text{mod }_2(x)}{2x} \geq 0$. The inequality is strict since it is not possible to have a cycle consisting entirely of even positive integers. The derivative of $T(x)$ is the following.

$$\mathrm{T}'(x) = \frac{\text{mod}'_2(x)}{2}\left(1 + x\ln(3)3^{\text{ mod }_2(x)}\right) + \frac{1}{2}3^{\text{ mod }_2(x)} \tag{6}$$

Note that

$$\text{mod}'_2(x) = \pi\sin(\frac{\pi x}{2})\cos(\frac{\pi x}{2}) \tag{7}$$

is zero at every integer. Thus, we see by Equations (5) and (6) that

$$\prod_{x\in\Omega} \mathrm{T}'(x) = \prod_{x\in\Omega} \frac{1}{2}3^{\text{ mod }_2(x)} < 1 \tag{8}$$

Hence the cycle is attractive which completes the proof.

Now we state the remainder representation theorem from Terras [13].

Theorem 2 *(The Remainder Representation Theorem of Terras). Let* $x_k = T^k(x_0)$ *be the* k^{th} *iterate of* $T(x)$ *on* $x_0 = x$ *and let* $z_k = \text{mod }_2(x_k)$. *Next let*

$$\lambda_k = \frac{3^{z_0+z_1+\cdots+z_{k-1}}}{2^k} \tag{9}$$

and let

$$\rho_k = \left(\frac{\lambda_k}{2}\right)\left(\frac{z_0}{\lambda_1} + \frac{z_1}{\lambda_2} + \cdots + \frac{z_{k-1}}{\lambda_k}\right). \tag{10}$$

Then,

$$T^k(x) = \lambda_k x + \rho_k. \tag{11}$$

Notice that the expansion factor we found for a positive integer cycle is precisely the coefficient function on the cycle. That is,

$$\prod_{x \in \Omega} \frac{1}{2} 3^{\bmod_2(x)} = \lambda_k. \tag{12}$$

Thus we obtain the following fact.

Proposition 3 *If Ω denotes a cycle of positive integers of length k and $x \in \Omega$, then λ_k is the expansion factor of Ω as a cycle of $T(x)$.*

The *stopping time* of $T(x)$ on an integer (or real) x is defined to be the smallest k, if it exists, such that $T^k(x) < x$; otherwise the *stopping time* is ∞. Notice that if $\lambda_k \geq 1$, then by the remainder representation theorem, $T(x) > x$. Thus, $\lambda_k < 1$ must occur before $T(x) < x$. This motivates the definition of the *coefficient stopping time* as the smallest $k \geq 1$ such that $\lambda_k < 1$ or ∞ if there is no such k. The above remark is that the coefficient stopping time is less than or equal to the stopping time. Remarkably, it appears that the converse holds as well. The coefficient stopping time conjecture is that the stopping time is equal to the coefficient stopping time for all integers $n \geq 2$.

The fact that Proposition (3) relates the classical coefficient function of Terras with the expansion factor of any positive cycle of integers is evidence that $T(x)$ is a natural generalization of $t(x)$ in the sense that the real dynamics of $T(x)$ are directly related to the behavior of classically studied functions.

In the next section we continue to develop facts about the real dynamics of $T(x)$. In particular, we show that $T(x)$ has negative Schwarzian derivative.

3 The Negative Schwarzian

Singer [12] established that iterates of functions on an interval with negative Schwarzian derivative have dynamics satisfying some nice properties. The Schwarzian derivative of $f(x)$ is denoted $Sf(x)$ and it is defined as follows.

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \tag{13}$$

Following [1] and [2] we state the following version of Singer's Theorem.

Theorem 4 *(Singer). If $f : I \to I$ is a C^3 map with negative Schwarzian derivative, then*
1. the immediate basin of any attracting periodic orbit contains either a critical point of $f(x)$ or a boundary point of I.
2. each neutral periodic point is attracting
3. there exists no interval of periodic points

term	parity
$E_1 = -3\,\pi^2 \cos^4(x)$	$-$
$E_2 = 4\,3^{2\sin^2(x)+1} \cos^2(x)\,\ln(3)$	$+$
$E_3 = -4\,3^{\sin^2(x)+1}\,\pi\,x \cos^4(x)\,\ln(3)$	$-$
$E_4 = -4\,3^{2\sin^2(x)+1}\,x^2 \cos^4(x) \ln^2(3)$	$-$
$E_5 = -8\,3^{\sin^2(x)}\,\pi\,\cos(x)\,\sin(x)$	$\pm$
$E_6 = -16\,3^{2\sin^2(x)}\,x\,\cos(x)\,\ln(3)\,\sin(x)$	$\pm$
$E_7 = -4\,3^{\sin^2(x)+1}\,\pi \cos^3(x)\,\ln(3)\,\sin(x)$	$\pm$
$E_8 = -2\,\pi^2 \cos^2(x) \sin^2(x)$	$-$
$E_9 = -4\,3^{2\sin^2(x)+1}\,\ln(3) \sin^2(x)$	$-$
$E_{10} = -8\,3^{\sin^2(x)}\,\pi\,x \cos^2(x)\,\ln(3) \sin^2(x)$	$-$
$E_{11} = -8\,3^{2\sin^2(x)+1} \cos^2(x) \ln^2(3) \sin^2(x)$	$-$
$E_{12} = -8\,3^{2\sin^2(x)}\,x^2 \cos^2(x) \ln^2(3) \sin^2(x)$	$-$
$E_{13} = 4\,3^{\sin^2(x)+1}\,\pi\,\cos(x)\,\ln(3) \sin^3(x)$	$\pm$
$E_{14} = 8\,3^{\sin^2(x)+1}\,\pi \cos^3(x) \ln^2(3) \sin^3(x)$	$\pm$
$E_{15} = -32\,3^{2\sin^2(x)}\,x \cos^3(x) \ln^3(3) \sin^3(x)$	$\pm$
$E_{16} = -3\,\pi^2 \sin^4(x)$	$-$
$E_{17} = -4\,3^{\sin^2(x)+1}\,\pi\,x\,\ln(3) \sin^4(x)$	$-$
$E_{18} = -4\,3^{2\sin^2(x)+1}\,x^2 \ln^2(3) \sin^4(x)$	$-$
$E_{19} = 16\,3^{\sin^2(x)}\,\pi\,x \cos^4(x) \ln^3(3) \sin^4(x)$	$+$
$E_{20} = -16\,3^{2\sin^2(x)}\,x^2 \cos^4(x) \ln^4(3) \sin^4(x)$	$-$

Table 1: *Terms of the Schwarzian Derivative*

We show that the Schwarzian derivative of the 3-power extention of the $3x+1$ function is nonnegative.

Theorem 5 *The Schwarzian derivative of $T(x)$ is negative for all $x \geq 0$.*

Proof. Note that at the critical points, this means the Schwarzian derivative is negative infinity. The proof is elementary although there are many details to verify. We first observe that if k is a constant, then $Sf(kx) = k^2 Sf(x)|_{x \mapsto kx}$. Therefore, it suffices to show that $\hat{T}(x) = T(\frac{2}{\pi}x)$ has negative Schwarzian derivative. We will do that by verifying that $2\pi^2(\hat{T}'(x))^2 S\hat{T}(x)$ is negative for $x \geq 0$. Direct computation of $2\pi^2(\hat{T}'(x))^2 S\hat{T}(x)$ leads to the twenty terms in Table 1. In each case we informally indicate a parity of $+$, $-$, or $\pm$ when the term is clearly nonnegative, nonpositive or changes sign for $x \geq 0$. Notice that only two of the terms are nonnegative although six terms change sign. The powers of sin, cos and ln appearing in the table are ordinary powers and not function iteration. We collect terms into the following six groups and then show the first is strictly negative and the others are nonpositive.
(A) $E_1 + E_2 + E_9$

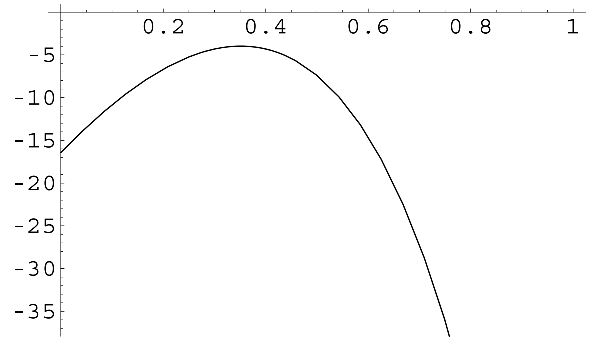

Figure 2: The Function $f_A(w)$.

(B) $E_{10}+E_{19}$
(C) $\frac{1}{2}E_{11}+E_{13}+E_{16}$
(D) $E_8+\frac{1}{2}E_{11}+E_{12}+E_{14}+E_{15}$
(E) $E_3+E_4+E_5+E_6+E_7+E_{17}+E_{18}$
(F) E_{20}
Group (A). Simplifying $f_A = E_1+E_2+E_9$ with the identity $\cos^2(x) = 1-\sin^2(x)$ and substituting $w=\sin^2(x)$ gives the function

$$f_A(w) = -3\,\pi^2\,(1-w)^2 + 4\,3^{2\,w+1}\,\ln(3)\,(1-w) - 4\,3^{2\,w+1}\,w\,\ln(3).$$

Figure 2 shows $f_A(w)$. Numerical calculus verifies that $f_A(w)\le -3.975$ for $0\le w\le 1$ and hence $E_1+E_2+E_9<0$ for all x.

Group (B). Direct computation shows that

$$E_{10}+E_{19} = 8\,3^{\sin^2(x)}\,\pi\,x\cos^2(x)\,\ln(3)\sin^2(x)\left(2\cos^2(x)\ln^2(3)\sin^2(x)-1\right).$$

Showing that the last factor is nonpositive is equivalent to checking that $\sin^2(2x)\le\frac{2}{\ln^2(3)}\approx 1.657$, which is true, and hence $E_{10}+E_{19}\le 0$.
Group (C). Simplifying we see

$$\frac{1}{2}E_{11}+E_{13}+E_{16} = -3\sin^2(x)\left(\pi\,\sin(x) - 2\,3^{\sin^2(x)}\,\cos(x)\,\ln(3)\right)^2$$

which is nonpositive.
Group (D). We see each term has a factor of $-2\cos^2(x)\sin^2(x)$ and hence we need to show that following five terms are positive.

$$\begin{aligned} f_D(x) = \big(&4\,3^{2\sin^2(x)}\ln^2(3)\,x^2 + 16\,3^{2\sin^2(x)}\,\cos(x)\ln^3(3)\,\sin(x)\,x \\ &-4\,3^{\sin^2(x)+1}\,\pi\,\cos(x)\ln^2(3)\,\sin(x) + 2\,3^{2\sin^2(x)+1}\ln^2(3)+\pi^2\big) \end{aligned} \quad (14)$$

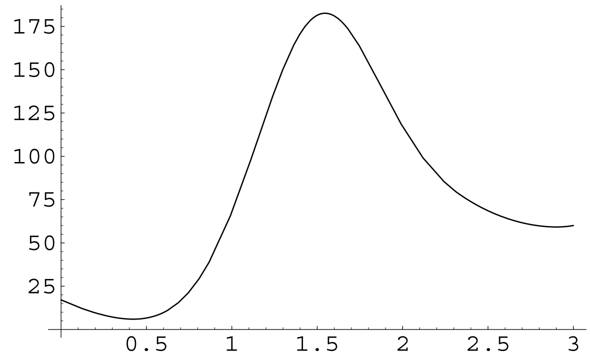

Figure 3: The Function $f_D(x)$ is positive on $[0,3]$.

Only two terms have mixed signs and they can be dominated by the first term for sufficiently large x. Note that the $3/4^{ths}$ the first term is larger than the second term when $x > \frac{8}{3}\ln(3) \approx 2.929632$ and $1/4^{th}$ the first term is larger than the third term when $x > \sqrt{2\pi} \approx 2.506$. Thus, the claim is true for $x > 3$. Figure 3 shows the above five terms on $[0,3]$. Numerical calculus verifies the function is positive on this range as required to complete this group.

Group (E). First notice that each term is nonpositive in the first quadrant, so we may assume that $x \geq \pi/2$. Next, each term has a factor of $-4\,3^{\sin^2(x)}$, so it suffices to show that the sum of terms, divided by that factor, is nonnegative. Moving those terms with odd powers of sine and cosine to one side means we need to show that for $x \geq \pi/2$, the following holds.

$$\begin{aligned}&\left(3^{\sin^2(x)}\,2\ln(3)\,x + \pi + \frac{3}{2}\pi\ln(3)\cos^2(x)\right)\,\sin(2x)\\&\qquad\leq 3\,x\,\ln(3)\left(3^{\sin^2(x)}\,\ln(3)\,x+\pi\right)\left(\cos^4(x)+\sin^4(x)\right)\end{aligned}\tag{15}$$

Now for $x \geq \pi/2$,

$$\left|\frac{3}{2}\pi\ln(3)\cos^2(x)\right| = \frac{5}{4}\pi\ln(3) + \frac{1}{4}\pi\ln(3) \leq \frac{5}{4}\ln(3)\pi + \frac{1}{2}x\ln(3)3^{\sin^2(x)}$$

The left-hand-side of Equation (15) is bounded as follows.

$$\begin{aligned}&\left|\left(3^{\sin^2(x)}2\ln(3)x + \pi + \frac{3}{2}\pi\ln(3)\cos^2(x)\right)\sin(2x)\right|\\&\qquad\leq \max(2.5, 1+\frac{5}{4}\ln(3))\left(3^{\sin^2(x)}\,\ln(3)\,x+\pi\right)\end{aligned}$$

Moreover, $\max(2.5, 1+\frac{5}{4}\ln(3)) = \max(2.5, 2.3732\ldots) = 2.5$. The right-hand-side of Equation (15) satisfies

$$3\,x\,\ln(3)\left(3^{\sin^2(x)}\,\ln(3)\,x+\pi\right)\left(\cos^4(x)+\sin^4(x)\right)$$

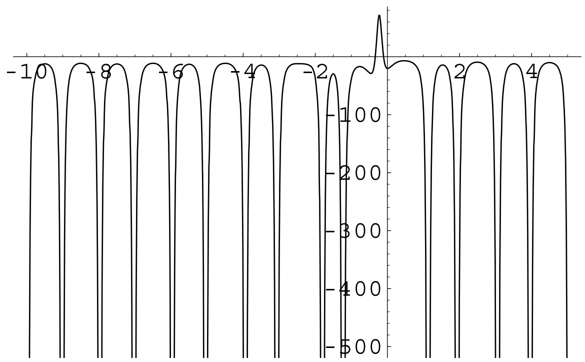

Figure 4: The Schwarzian Derivative of $T(x)$.

$$\geq 3\,(\pi/2)\,\ln(3)\left(3^{\sin^2(x)}\,\ln(3)\,x + \pi\right)(1/2)$$

since $x \geq \pi/2$ and $\left(\cos^4(x)+\sin^4(x)\right) \geq 1/2$. Notice that $3\,(\pi/2)\,\ln(3)(1/2) \approx 2.58854 \geq 2.5$ which implies the desired inequality. Thus Equation (15) holds and hence this group is nonpositive.
Group (F). This term is always nonpositive.

Our theorem established that the Schwarzian derivative is negative for $x \geq 0$ which is the classical region of interest for the $3x + 1$ function. However, it appears that the Schwarzian derivative is negative except for a short interval contained in $[-0.4, -0.1]$. See Figure 4. While we expect that arguments like those used in the proof of Theorem 5 could be used for x sufficiently negative, we have not carried out the details. We have checked that some of the groups (A)-(F) give positive values as $x \to -\infty$ and hence some other organization would be needed.

4 Fixed Points and Critical Points

In this section we consider the nonnegative fixed and critical points of $T(x)$. In general there is a critical point "near" to each integer and a fixed point in between. Two invariant intervals between fixed points occur and these have simple dynamics. We are able to establish approximate positions for all the nonnegative fixed points and critical points.

Our first theorem establishes the regular appearance and approximate position of positive fixed points and critical points.

Theorem 6 *Let the constant δ be defined by $\delta = \frac{2}{\pi}\sin^{-1}\sqrt{\log_3(2)} \approx 0.584336$. The nonnegative fixed points of $T(x)$ are $0 = \mu_0 < \mu_1 < \mu_2 < \ldots < \mu_n < \ldots$. They satisfy $n - 1 \leq \mu_n \leq n$ and for large n, they are close to being an even* *integer $\pm\delta$. In particular,*

$$\mu_n = \begin{cases} n - \delta + O\left(\frac{1}{n}\right) & \text{if } n \text{ is even} \\ n - 1 + \delta + O\left(\frac{1}{n}\right) & \text{if } n \text{ is odd} \end{cases} \tag{16}$$

Furthermore, the nonnegative critical points of $T(x)$ are $c_1 < c_2 < \ldots < c_n < \ldots$ and they satisfy $\mu_n \le c_n \le \mu_{n+1}$. In particular,

$$c_n = n - (-1)^n \frac{2}{\pi^2 \ln(3)} \frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{17}$$

Proof. We may establish the basic pattern of oscillations by noting that the function

$$h(x) = \frac{T(x) - x/2}{x + 1/2}$$

is zero at the even integers and one at the odd integers. Furthermore, its derivative is zero exactly at the integers for $x \ge 0$. That fact may be carefully shown by analyzing terms in the style of the proof of Theorem 5. The fixed points of $T(x)$ correspond to intersections of $h(x)$ with $\frac{x}{2x+1}$ which monotonically increases to the asymptote $y = 1/2$ as $x \to \infty$. Hence there are no extra positive critical points or fixed points.

We turn to the more precise placement of the fixed points and critical points of $T(x)$.

The fixed points of $T(x)$ satisfy

$$3^{\operatorname{mod}_2(\mu_n)} - 2 = \frac{-\operatorname{mod}_2(\mu_n)}{\mu_n} \tag{18}$$

Let z_n be zero or one according to the parity of n and let ϵ_n denote the error of approximation in Equation (16). That is, $\mu_n = n - z_n \pm \delta + \epsilon_n$. Notice that $\mu_n \sim n$ and hence the right-hand-side of Equation (18) is $O\left(\frac{1}{n}\right)$. For even n, trigonometric identities imply that $\operatorname{mod}_2(x+n) = \sin^2\left(\frac{\pi(x+n)}{2}\right) = \operatorname{mod}_2(x)$. Of course, $n - z_n$ is even for all n. Also, δ was defined so that the following power series holds.

$$3^{\operatorname{mod}_2(x\pm\delta)} - 2 = \pm Cx + O\left(x^2\right) \tag{19}$$

where $C \approx 3.33096$ is an exactly computable constant. Hence,

$$\begin{aligned} 3^{\operatorname{mod}_2(\mu_n)} - 2 &= 3^{\operatorname{mod}_2(n - z_n \pm\delta + \epsilon_n)} - 2 = 3^{\operatorname{mod}_2(\pm\delta + \epsilon_n)} - 2 \\ &= \pm C\epsilon_n + O\left(\epsilon_n^2\right) \end{aligned} \tag{20}$$

Using Equation 18 and the above remark about it, we see the error ϵ_n is $O\left(\frac{1}{n}\right)$ as claimed.

Next consider the critical points. For even integers, n, we know that $\operatorname{mod}_2(x+n) = \operatorname{mod}_2(x)$. Recalling Equation (7), we see likewise for the

n	μ_n	c_n
1	0.31581620327986408296	1.1309941431270213886
2	1.5155526112246360140	1.9344565225151632834
3	2.5189777540293784504	3.0551547747623897629
4	3.4649453878164739882	3.9620246295489206948
5	4.5458698070488534748	5.0346163272795386811
6	5.4481739006727628044	5.9731645377030619937
7	6.5570236427362714510	7.0251934109819066797
8	7.4398921913666236970	7.9792331855813816122
9	8.5631514104741717243	9.0197963599168494544
10	9.4349665507394106536	9.9830581933811043246
⋮	⋮	⋮
100	99.417562309023307386	99.998172064566257856
101	100.58245959056682904	101.00182077495494432
⋮	⋮	⋮
1000	999.41585374523525213	999.99981571466777848
1001	1000.5841464758883631	1001.0001842129025298

Table 2: Some Approximate Fixed Points and Critical Points of $T(x)$

derivative: $\text{mod}_2'(x+n) = \text{mod}_2'(x)$. Thus, if we use those facts upon substituting $c_n = n + \epsilon_n$ into $T'(c_n) = 0$ and recalling Equation (6) which gives $T'(x)$, we obtain the following equation

$$3^{\text{ mod }_2(\epsilon_n)} + (n+\epsilon_n)3^{\text{ mod }_2(\epsilon_n)} \ln(3) \text{ mod }_2'(\epsilon_n) + \text{mod}_2'(\epsilon_n) = 0 \qquad (21)$$

Now expanding Equation (21) in a power series we see that $\epsilon_n = -\frac{2}{\pi \ln(3)} \frac{1}{n} + O\left(\frac{1}{n^2}\right)$.

When n is odd, $\text{mod}_2(x+n) = 1 - \text{mod}_2(x)$ and $\text{mod}_2'(x+n) = -\text{ mod }_2'(x)$, and the power series expansion changes the sign of the $O\left(\frac{1}{n}\right)$ term, giving the result.

Table 2 shows some numeric approximations to some fixed points and critical points. Notice that the critical points are close to the corresponding integers and the difference, as expected in light of Theorem 6, is $O\left(\frac{1}{n}\right)$. Next, the fixed points are in between, but these are close to the integer $\pm\delta$.

We have the following results regarding invariant intervals.

Theorem 7 *The interval* $[0, \mu_1]$ *is an invariant set and every point in* $[0, \mu 1)$ *is attracted to the fixed point* 0.

Proof. Note that $[0, \mu 1]$ is an invariant set with $T'(0) = 0.5$ and $T'(\mu_1) \approx 1.59$. Thus, 0 is an attracting fixed point and μ_1 is a repelling fixed point. On $[0, \mu 1)$ we have $0 \leq T(x) < x$ and no other fixed points, hence iteration of $T(x)$ on points from that interval converges to 0.

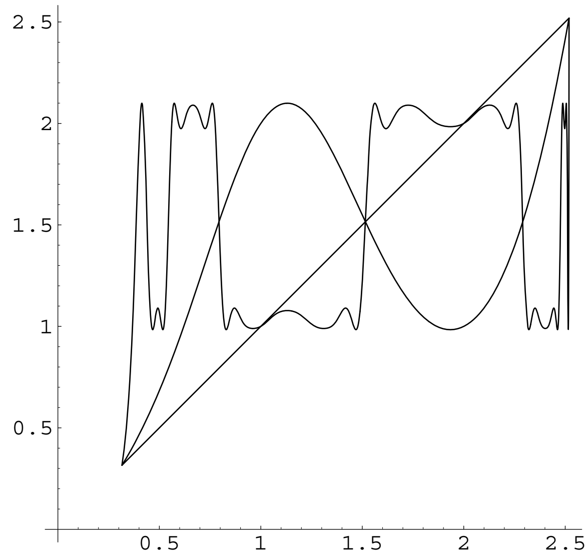

Figure 5: The functions x, $T(x)$, and $T^4(x)$ on $[\mu_1, \mu_3]$.

Theorem 8 *The interval* $[\mu_1, \mu_3]$ *is an invariant set and every point in* (μ_1, μ_3) *is attracted to the cycle* $(1, 2)$.

Proof. This interval contains the attractive 2-cycle $(1, 2)$ and the repelling fixed points μ_1, μ_2 and μ_3. Figure 5 shows the fourth iterate of $T^4(x)$ on this interval. Notice are no 4-cycles. In light of Sarkovskii's Theorem [3], there are no other periodic points. The two critical points of $T(x)$ on this interval are attracted to the cycle $(1, 2)$. Thus, every point in the interior of this interval is attracted to the cycle $(1, 2)$.

The behavior of $T(x)$ on $[\mu_1, \mu_3]$ contrasts with the behavior of $C(x)$ on its corresponding interval. The function $C(x)$ has two attractive 2-cycles with the basins of attraction forming Cantor sets. We will see that the simpler behavior of $T(x)$ allows us to give a natural definition for the total stopping time for $T(x)$.

The behavior of $T(x)$ on (μ_3, ∞) is much more complicated than for $0 \leq x \leq \mu_3$. For example, Figure 6 illustrates that there are several repelling three cycles on $[\mu_3, \mu_5]$. In light of Sarkovskii's Theorem [3], that implies periodic points of all orders which on an interval implies chaos [9]. This is not surprising given the difficult to analyze, random nature that $T(x)$ appears to have.

Table 3 shows the complex cycles of $T(x)$ that were dynamically found by the basin of attraction algorithm used in [4] for images showing $-6 \leq Re(x) \leq 10$. Notice that it found only the one attractive positive cycle, although there

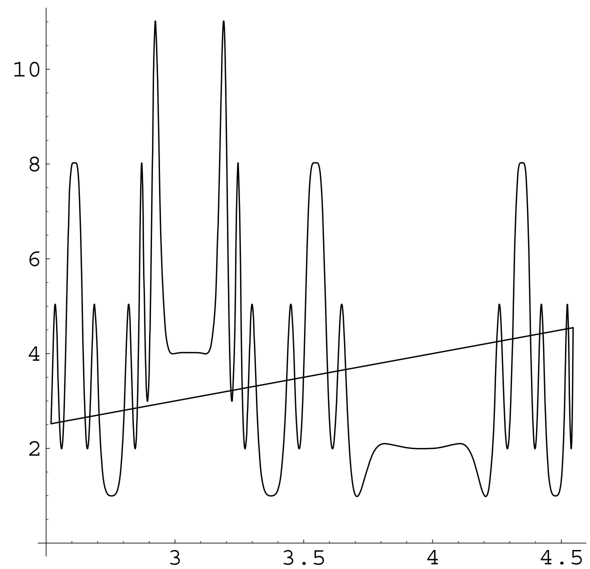

Figure 6: The functions x and $T^3(x)$ on $[\mu_3, \mu_5]$.

0.5	(0)
0.469947	(-1.15387)
1.5	(-1)
0.75	(1, 2)
0.703092	(-10.0157, -5.01091, -7.01408)
1.12501	(-10, -5, -7)

Table 3: Some Cycles for $T(x)$.

are some of the expected negative cycles including an attractive one that shadows the cycle $(-10, -5, -7)$ and a shadow of the fixed point (-1).

The situation is similar for $C(x)$, see Table 4. However, there is also a cycle shadowing the cycle $(1, 2)$, and the 11-cycle $(-136, -68, ... - 61, -91)$ and its shadow were also observed. In particular, the dynamics of $C(x)$ appears more complicated than those for $T(x)$ because of the extra attractive cycles, especially the positive one.

Chamberland's Theorem 6.1, see [1], shows that any continuous interpolation of $t(x)$ on $x > 0$ has a three cycle, a homoclinic orbit and a divergent trajectory. Thus, our function $T(x)$ must have all those properties. However, in the next section we conjecture that the critical points near the odd integers are "well placed". Before turning to that section, we empirically observe the placement of complex fixed points and critical points for $T(x)$.

0.5	(0)
0.385708	(-1.27773)
1.5	(-1)
0.75	(1, 2)
-0.230754	(1.19253, 2.13866)
0.0363716	(-10.0349, -5.046, -7.04531)
1.12504	(-10, -5, -7)
0.0035933	(-136.002, -68.0033, -34.0035, -17.0027, -25.0038, -37.0048, -55.0051, -82.0042, -41.0056, -61.0052, -91.0038)
1.08086	(-136, -68, -34, -17, -25, -37, -55, -82, -41, -61, -91)

Table 4: Some Cycles for $C(x)$.

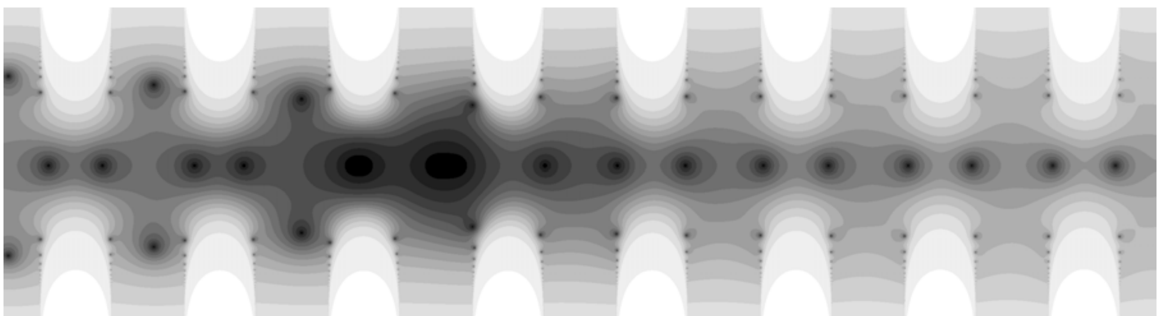

Figure 7: The Contours of $|T(x) - x|$ Showing Complex Fixed Points for $-6 \leq Re(x) \leq 10$

Figure 7 shows levels of $|T(x) - x|$ in the complex plane with $-6 \leq Re(x) \leq 10$ and $-2.2 \leq Im(x) \leq 2.2$. The regions around the fixed points are shown in black and we see the fixed points along the real axis, expected in light of Theorem 6. However, there also appear to be many infinite families of fixed points with increasing imaginary part. Experiments with Newton's method confirm the exisitence of these fixed points with nonzero imaginary part.

Figure 8 shows contour levels for $|T'(x)|$ viewed as a function of a complex variable with $-6 \leq Re(x) \leq 10$ and $-2.2 \leq Im(x) \leq 2.2$. The dark regions are lowest and show the pattern of critical points in the complex plane. Again there appear to be many infinite families of critical points with increasing magnitude imaginary part.

The placement of the complex fixed points and critical points is consistent with the small isolated regions with small stopping times observed away from the real axis in [4].

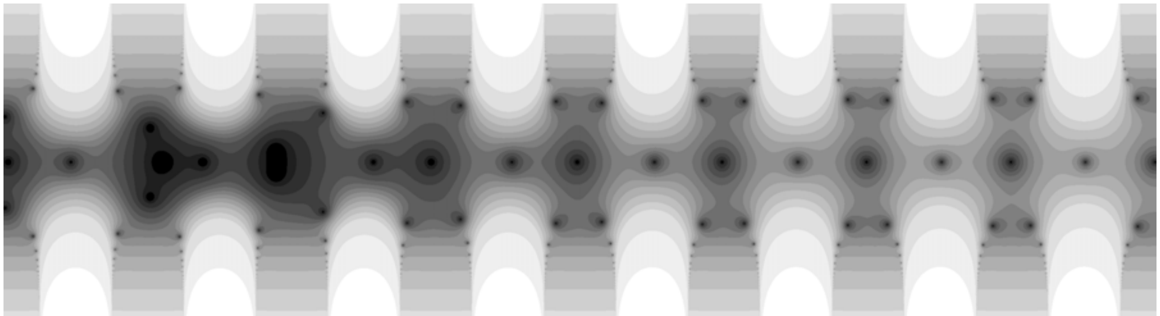

Figure 8: The Contours of $|T'(x)|$ Showing Complex Critical Points for $-6 \leq Re(x) \leq 10$

5 Odd Critical Point Conjecture

Recall that the total stopping time of a positive integer x is defined to be the smallest k such that $t^k(x) = 1$ if it exists and it is ∞ if there is no such k. When generalizing this to real x, one faces the difficulty that values attracted to the cycle $(1, 2)$ typically do not reach it. One can require that an iterate come sufficiently close to 1 as the condition, [4], but the choice of comparison tolerance is arbitrary. However, in light of Theorem 8 we see there is a natural choice for the total stopping time of $T(x)$. Once an iterate enters the interval (μ_1, μ_3), unless it hits the fixed point μ_2, we have seen that it is attracted to the cycle $(1, 2)$. That interval is naturally split by the repelling fixed point μ_2. Thus, we define the *total stopping time* of a real number x to be the least k, if it exists, such that $\mu_1 < T^k(x) < \mu_2$ and it is ∞ if there is no such k.

Note that we have seen that points in $[0, \mu_1)$ are attracted to 0, and hence would have infinite total stopping time. Also, the fixed points μ_k all have infinite total stopping time. Moreover, we noted that $T(x)$ must have divergent trajectories. All the points in those trajectories must have infinite total stopping time. Thus there is no hope of claiming that the total stopping time is finite for positive x. We will see there is evidence that the total stopping time is the same for the odd positive integers as for the associated critical points.

Consider even and odd examples. Table 5 shows selected iterates of $T(x)$ on 27 and the nearby critical point c_{27}. The iterates on c_{27} are of course approximations. The entries here are truncations of 200 digit computations. Notice that the iterates of the critical point stay close to the iterates of 27. Figure 9 shows the total stopping time on a region in the complex plane around 27 and c_{27}. The image center is at 27 and it is marked with a black hash mark from above. The critical point is two thirds of the way from the center toward to the right edge and it is also marked with a hash mark. The

k	$T^k(27)$	$T^k(c_{27})$
0	27	27.0067545035
1	41	41.0050660042
2	62	62.0032885663
3	31	31.0025664593
4	47	47.0030112618
⋮	⋮	⋮
44	3077	3077.0000529260
45	4616	4616.0000443393
46	2308	2308.0000344719
47	1154	1154.0000209546
48	577	577.0000111646
⋮	⋮	⋮
66	5	5.0000000973
67	8	8.0000001460
68	4	4.0000000730
69	2	2.0000000365
70	1	1.0000000183

Table 5: Iterates on 27 and c_{27}.

regions with total stopping time 70 are shown in cyan. Other finite stopping times appear in red. Notice that the immediate basin of total stopping time 70 around the point 27 appears to have 27 near the left edge and the critical point appears to be in its center. Zooms of a couple orders of magnitude show 27 to be an interior point of the basin even though it is very close to the edge.

Table 6 shows selected iterates of $T(x)$ on 54 and the nearby critical point c_{54}. Notice that the iterates of the critical point diverge from those of 54 at around the tenth iterate. Figure 10 shows the total stopping time on a region in the complex plane around 54 and c_{54}. The center is 54 and the hash marks point to the points c_{54} and 54. Notice that 54 is still on the left edge of the immediate basin of 54 with total stopping time 71. However, the critical point is to the left and apparently not near that basin.

While this type of failure of a critical point to shadow the nearby integer seems slightly atypical (but not rare), we know of no examples where an odd critical point fails to shadow the integer. We have checked that the total stopping time all odd integers less than $180,000$ matches the total stopping time for the associated critical point.

Lagarias [6] mentions that V. Vyssotsky found

$$n_V = 37\ 66497\ 18609\ 59140\ 59576\ 52867\ 40059$$

k	$T^k(54)$	$T^k(c_{54})$
0	54	53.9966405567
1	27	26.9991601600
2	41	40.9986619383
3	62	61.9976922320
4	31	30.9993002093
5	47	46.9988879845
6	71	70.9980941404
7	107	106.9960881520
8	161	160.9874561304
9	242	241.8780269644
10	121	125.8722964761
11	182	65.7618391600
12	91	38.1435957646
13	137	20.1741877894
14	206	10.9660735658
⋮	⋮	⋮
26	334	4.4900479607
27	167	4.0644796600
28	251	2.0603056805
29	377	1.0448012336
30	566	2.0562399989

Table 6: Iterates on 54 and c_{54}.

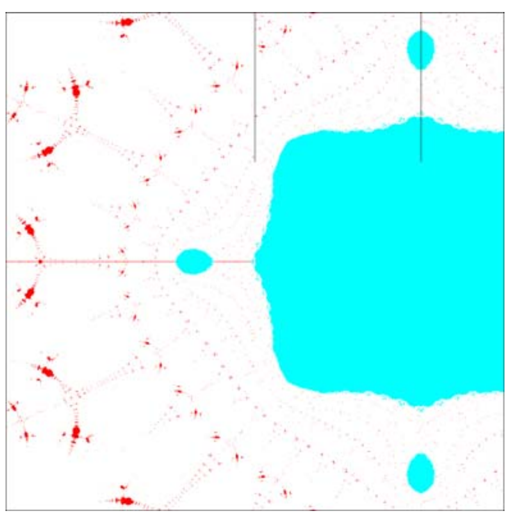

Figure 9: The Total Stopping Time of $T(x)$ Near 27.

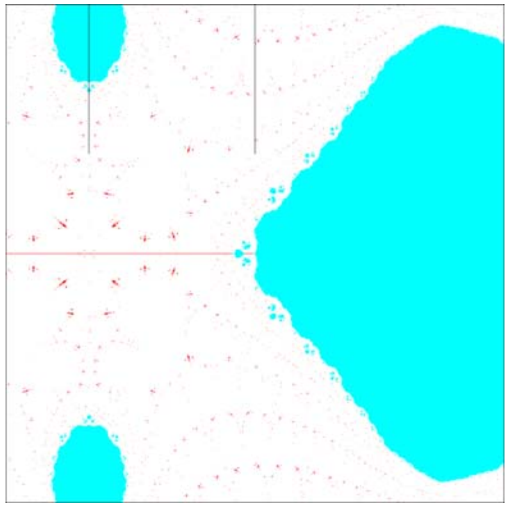

Figure 10: The Total Stopping Time of $T(x)$ Near 54.

has an unusually large total stopping time, 2565. We computed the critical point c_{n_V} using high precision floating point arithmetic in *Mathematica* [10]. Using Newton's method on the estimate from Equation (17) we computed the critical point to 50,000 decimal places. We confirmed that the total stopping time for c_{n_V} was also 2565. During those steps, the floating point precision was reduced to $28,147$ digits. The value of $T^{2565}(c_{n_V}) \approx 1 + 7.1\,10^{-72}$.

Conjecture 1 *(Odd Critical Point Conjecture) Let* n *be an odd, positive integer and* c_n *be the associated critical point of* $T(x)$*.*
(i) The point c_n *is attracted to the cycle* $(1, 2)$*.*
(ii) The total stopping time of n *is the same as the total the stopping time of* c_n*.*
(iii) c_n *is in the immediate basin of total stopping time equal to the value at* n*.*

The Odd Critical Point Conjecture implies that the $3x + 1$ function has no positive integer cycles except $(1, 2)$. That follows since all odd integers would follow their critical points to the cycle $(1, 2)$.

6 Conclusion

We have seen that the function $T(x)$ generalizes the $3x + 1$ function. The real dynamics of $T(x)$ include the fact that every positive integer cycle of $T(x)$ must be an attractive cycle with expansion factor given by the Terras coefficient function on the cycle. The Schwarzian derivative is negative for $x \geq 0$. We have seen that the fixed points and critical points of $T(x)$ can be approximately located and two invariant intervals appear. The simple dynamics on the interval containing the cycle $(1, 2)$ allows a natural definition to be given for the total stopping time for $T(x)$. We saw the odd critical points, unlike the even ones, appear to be closely bound to their associated critical points, yielding the "odd critical point conjecture". That conjecture implies there are no nontrivial positive cycles for the $3x + 1$ function.

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