ITERATED FUNCTION SYSTEMS WITH SYMMETRY IN THE HYPERBOLIC PLANE

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Abstract - Images are created using probabilistic iterated function systems that involve both affine transformations of the plane and isometries of hyperbolic geometry. Figures of attractors with striking hyperbolic symmetry are the result.

Key words: chaos, IFS, hyperbolic symmetry, Escher

1. INTRODUCTION

Hyperbolic symmetry is intriguing to many artists and mathematicians. Schattschneider describes an exchange of ideas on the subject between Coxeter, a mathematician, and Escher, an artist [1]. Coxeter's illustration of hyperbolic tilings in *A Symposium on Symmetry* sparked the interest of Escher, who had contributed two plane illustrations to the paper. Although he didn't understand the mathematical ideas behind the symmetries, Escher saw this hyperbolic tiling as a solution to creating a repeating motif which decreases in size from the center outwards. Escher went on to create patterns containing hyperbolic symmetries in his *Circle Limit I, II, III, IV* and in others. [1]

There has been much recent study of symmetric attractors created by functions and iterated function systems. Chaotic attractors containing the symmetries of the seventeen planar crystallographic groups are illustrated in [2]. Cyclic symmetries are evident in the figures of [3], [4] and [5]. In [6] circular patterns are created using graphics manipulations in conjunction with mathematical techniques. However, none of these produce Escher-like motifs which decrease in size as they move from the center.

Iterated function systems (lists of maps) are our primary tool in creating such motifs. When these maps are contractions there is a well developed theory of the associated attractors [7]. Remarkably, one may take ordinary photographic images and approximate them very well with iterated function systems. This results in impressive image compression [8]. These iterated function systems are also popular for creating fractals [3,7,9,10]. Even randomly selected affine transformations may result in intriguing and beautiful images [11].

In recent years, mathematicians have become interested in creating hyperbolic patterns using computers. Dunham outlines the creation of hyperbolic patterns in [12]. Chung, Chan, and Wang create escape time images with hyperbolic symmetry in [13]. We develop a method of creating iterated function systems with symmetries in the hyperbolic plane. In particular, we investigate the result of mixing the isometries of various discrete hyperbolic groups with ordinary affine transformations. This mixing of affine transformations with the highly structured symmetries of hyperbolic tilings results in a pleasing tension between randomness and structured patterns. By combining such transformations and symmetries, we create aesthetically pleasing illustrations in which the size of a cell in a circular pattern decreases as it moves outward.

2. TWO MODELS OF HYPERBOLIC GEOMETRY

Two widely used models for hyperbolic geometry are the Weierstrass and Poincare models. The Poincare model encloses the hyperbolic plane in the unit circle, *U.* In this model, hyperbolic lines are circular arcs or diameters which meet the unit circle at right angles. In this representation angles are preserved while distances are distorted. An *n*-sided polygon is depicted in the Poincare model by an area bounded by *n* of those circular arcs.

In contrast, the Weierstrass model, *W*, preserves distances as well as angles. This model consists of the surface of the upper half hyperboloid defined by $z = \sqrt{1 + x^2 + y^2}$. It is possible to convert points in the Weierstrass model to the Poincare model by a suitable projection. Adapting the notation in [12] and [13], this projection $g: W \to U$ is given by the definition:

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{1+z}\begin{pmatrix} x \\ y \end{pmatrix}$$

The inverse of *g* is given by:

$$g^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1 - x^2 - y^2}\begin{pmatrix} 2x \\ 2y \\ 1 + x^2 + y^2 \end{pmatrix}$$

It is straight forward to check that g^{-1} gives points on the upper half hyperboloid. While conducting our experiments it is convenient for us to make some computations in *W* and others in *U* while graphing all results in *U*.

3. SYMMETRY GROUPS IN THE HYPERBOLIC PLANE

Tilings of the hyperbolic plane with *p*-sided polygons meeting *q* at a vertex are denoted by (*p*,*q*). It is well known that there is a (*p*,*q*) tiling of the hyperbolic plane if and only if $(p-2)(q-2) > 4$, see [14]. An illustration of the tiling (5,4) can be found in Figure 1. As required, we have pentagons meeting four at each vertex. Note that the Poincare model distorts distances such that, as one looks toward the edge of the circle, the cells of the tiling seem to decrease rapidly in size. This model, with its radially decreasing cells, is the one which intrigued Escher.

In a given tiling (p,q), elements of associated symmetry groups can be generated by the following reflections illustrated in Figure 2 (where one cell of the (5,4) tiling is used as an example):

1. The reflection A across one edge of the p-sided polygon;
2. The vertical reflection B across the x-axis; and
3. The reflection C across a line from the origin to a vertex of the polygon.

As in [12] and [13], we represent the transformations with matrices A, B, and C acting on the upper half hyperboloid, W, in $\Re^3$ as follows:

$$A = \begin{pmatrix} -\cosh(2b) & 0 & \sinh(2b) \\ 0 & 1 & 0 \\ -\sinh(2b) & 0 & \cosh(2b) \end{pmatrix}, \text{ where } b = \cosh^{-1}\left(\frac{\cos\left(\frac{\pi}{q}\right)}{\sin\left(\frac{\pi}{p}\right)} \right),$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \cos\left(\frac{2\pi}{p}\right) & \sin\left(\frac{2\pi}{p}\right) & 0 \\ \sin\left(\frac{2\pi}{p}\right) & -\cos\left(\frac{2\pi}{p}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is also useful to express two more transformations for the formation of the symmetry groups. We define $S=CB$, a counter-clockwise rotation by $2\pi/p$ around the origin. Similarly, we define $T=AC$, a counter-clockwise rotation by $2\pi/q$ around a vertex of the central polygon of the tiling.

There are three classical discrete symmetry groups of the hyperbolic plane which we use. The most natural, denoted $[p,q]$, has p-fold central dihedral symmetry, reflections across the sides of the polygons, and q-fold rotations around the vertices of the polygons. This first group, $[p,q]$, can be generated by the symmetries S, T, and C. The second group, $[p,q]^+$, is the subgroup consisting of all compositions of an even number of reflections. It is generated by S and T, preserves orientation and has index 2 in $[p,q]$. The third subgroup, $[p^+, q]$, is generated by S and A and also has index 2 in $[p,q]$. In this group there are p-fold rotations around the center of the polygon and mirrors across the edges of the polygons in the tiling.

4. CLASSICAL ITERATED FUNCTION SYSTEMS

Classical iterated function systems utilize transformations of the plane. The skew Sierpinski triangle, see Figure 3, may be constructed using three affine transformations. The first transformation halves the values of both coordinates, the second adds (0, 0.5) to the result of the first transformation, and the third adds (0.5, 0) to the result of the first transformation. That is,

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5x \\ 0.5y \end{pmatrix},\ T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5x \\ 0.5y + 0.5 \end{pmatrix},\ \text{and}\ T_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5x + 0.5 \\ 0.5y \end{pmatrix}.$$

When these transformations are randomly applied to a point and the corresponding points are plotted, Figure 3 is the result. Here the image is on a scale with both coordinates varying from 0 to 1. Notice that the fractal in that image is preserved under the three transformations. In other words, if we shrink the entire image by half (that is, apply T_1), we obtain the lower left portion of the image; if we apply T_2 to the image, we obtain the upper left portion of the figure; if we apply T_3 to the image, we obtain the lower right portion of the figure. When these three portions are combined in a collage, we obtain the entire figure.

5. OUR EXPERIMENTS

Since we can easily switch between the Poincare and Weierstrass models, we may speak freely of both affine transformations and the transformations associated with hyperbolic symmetry acting on points in the hyperbolic plane. However, in practice we apply g and g^{-1} where necessary. In our experiments, we use one, two or three affine transformations and generators of the hyperbolic symmetry group to form an iterated function system. The hyperbolic isometries guarantee the symmetry while the affine maps provide some mixing.

For example, Figure 4 is an illustration of an attractor created with only one affine contraction that reduces distances. The symmetry group illustrated by this figure is $[4^+,6]$. The affine transformation used to create this image is

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.33x \\ 0.33y \end{pmatrix}.$$

Similar to the Sierpinski triangle, each cell of this attractor appears to be a rescaled version of the complete figure. It is particularly evident in this figure the if we apply an affine rescaling to the whole figure, we map the figure back onto its central circular cell. In addition, the hyperbolic transformations map a cell of the attractor onto another cell. Since our figure has $[p^+,q]$ symmetry, we have four-fold rotations around the centers of the circular cells as well as reflections across them. Observe the white space surrounded by six of the circular cells. The elementary nature of our affine transformation only rescales the figure; hence, the attractor contains extra symmetry. Notice that if a single affine contraction were iterated by itself in the Euclidean plane a single fixed point would result. However, the result is far richer with these mixed symmetries.

Figure 5 illustrates an attractor with the symmetries of the group [5,6]. Here, we have all possible reflections and rotations on the tiling (5,6). Notice the cluster of circles creating the central cell which contains five-fold dihedral symmetry. As a result of the overlapping central mirrors the central black space appears surrounded by 10 overlapping cells. One can further observe the symmetry by focusing on the six groups of clustered circles surrounding any one of the largest black spaces. Color here, as well as in all of our figures, is used to differentiate areas hit by our iterated function system at various frequencies.

The basic cell is far less obvious in Figure 6, an illustration of the symmetry group $[3,7]^+$. This symmetry can be easily seen by the three stars meeting at a common point. Note the seven-fold rotational symmetry of each. The actual vertices of the basic cell in this figure are the centers of the star-shaped objects, which the eye may interpret to be a pseudo-cell. Since the group $[3,7]^+$ preserves orientation, there are no mirrors in this image.

The attractor in Figure 7 was generated with an iterated function system containing two affine transformations. It illustrates the symmetries of the group $[6,4]^+$. Notice the six-fold symmetry of the central cell as well as the cluster of four such cells surrounding the white space. Once again, the figure has no reflectional symmetry. It is also interesting to point out that there are two distinct elements or images of the whole figure after one affine transformation because the iterated function system contains two such transformations.

Figure 8 illustrates symmetry group [5,4] and was generated with three randomly chosen affine transformations. In this experiment, we achieve a more chaotic aspect by choosing three affine transformations with coefficients between 0.3 and -0.3.

6. IMPLEMENTATION

Some comments about implementation are in order. The group $[p,q]^+$ is generated by *S* and *T*. However, if we randomly apply one of the affine transformations or the map induced by *S* or *T*, a very biased image with the desired symmetries is created. Before correcting the bias, points tended towards certain sections of the image much more frequently than aesthetically desirable. To provide the image with balance, we do two things. We apply many of the transformations of the hyperbolic plane, instead of a short list of generators. More specifically, we create a list of all the hyperbolic isometries of the form $S^i T^j S^k$ where $0 \le i < p$, $0 \le j < q$ and $0 \le k < p$. By randomly applying one of these hyperbolic isometries or one of the affine transformations composed with an isometry, we create an image that is much more balanced in a rotational sense. However, portions of the image near the edge are visited far too frequently. In response, we run a small experiment before iterating. We create a tiny square near the attractor (found by iterating just the affine part of the iterated function systems) and follow the image of the square under each matrix. Ergo, the probability that we select a given matrix is related to the area of the image of the square under the application of that matrix. Indubitably, this compensates to a large degree for the warping of distance and area in the Poincare model. By extending the list of hyperbolic isometries and choosing a bias based on the area of the image of the tiny square, we add much more balance to the image. Similar ideas are employed when creating images for the other groups. To generate images with [p,q] symmetry we apply matrices of the form $S^i C^j T^k S^m$ and to generate images with $[p^+,q]$ symmetry we use matrices of the form $A^i S^j A^k S^m$.

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APPENDIX. PSEUDOCODE FOR ITERATED FUNCTION SYSTEMS WITH HYPERBOLIC SYMMETRY.

Input p, q with $(p-2)(q-2)>4$
Let A, B, C, S, and T be the matrices as defined in Section 3
Let g and g^{-1} be as defined in Section 2.
Create a list G_s , of affine maps on $\Re^2$ that include the identity map
(if chosen at random, coefficients between ±0.3 seem good) .
Create a list H_t of hyperbolic isometries on W by taking all matrices of the following form
case (by symmetry group desired)
[p,q]+ : $S^i T^j S^k$, where $0 \le i < p$, $0 \le j < q$, and $0 \le k < p$
[p,q] : $S^i C^j T^k S^m$, $0 \le i < p$, $0 \le j < 1$, $0 \le k < q$, and $0 \le m < p$

[p$^+$,q] : $A^i S^j A^k S^m$, $0 \le i < 1$, $0 \le j < p$, $0 \le k < 1$, and $0 \le m < p$
end case

Select a starting point, for example $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$.

Let $k = 0$
While k < 10,000,000 or so do

Pick s at random from the legal indices to G_s

Let $\begin{pmatrix} x \\ y \end{pmatrix} = G_s \begin{pmatrix} x \\ y \end{pmatrix}$

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = g^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$

Pick t at random from the legal indices to H_t

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = H_t \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Let $\begin{pmatrix} x \\ y \end{pmatrix} = g \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

If $k > 1000$ increment the frequency count for the pixel corresponding to position (x,y)
$k = k + 1$
End while
Choose a palette
Optimize the contrast in the frequency count array
Plot the result

Figures

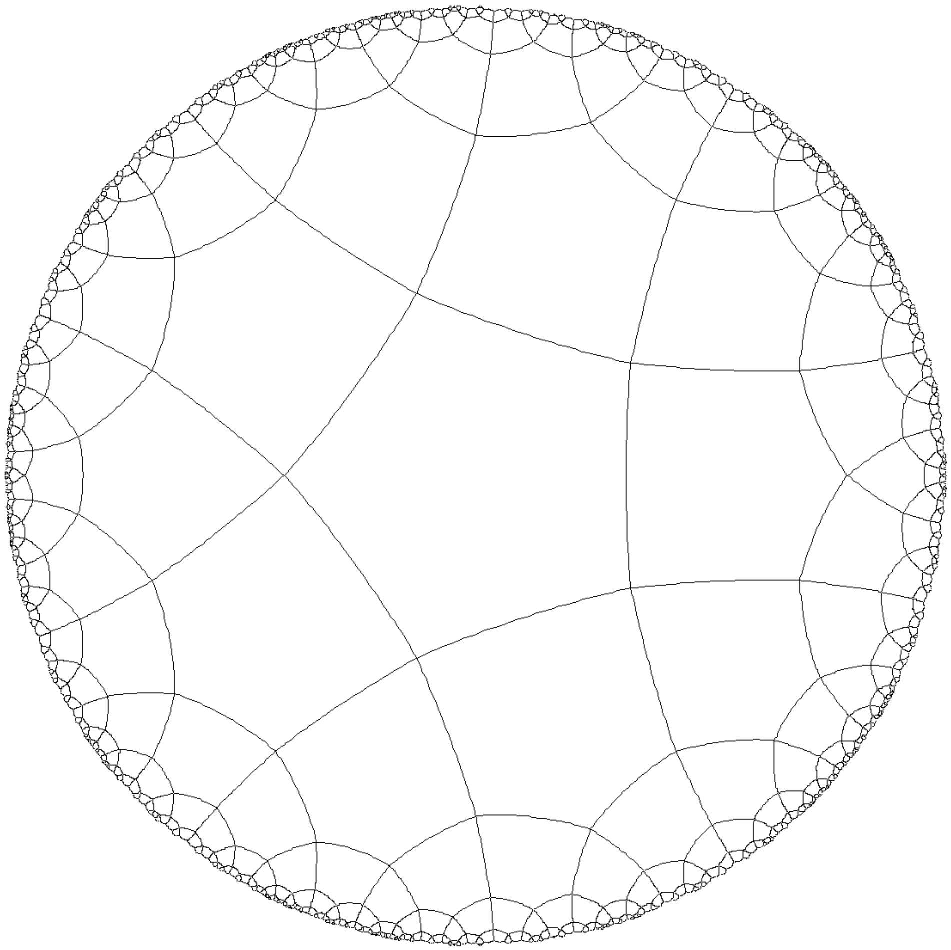

Fig. 1. A regular (5,4) tiling of the hyperbolic plane.

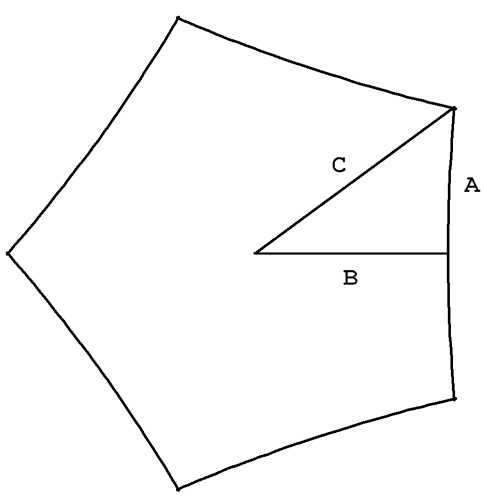

Fig. 2. A polygon from the (5,4) tiling with *A*, *B*, and *C* reflections.

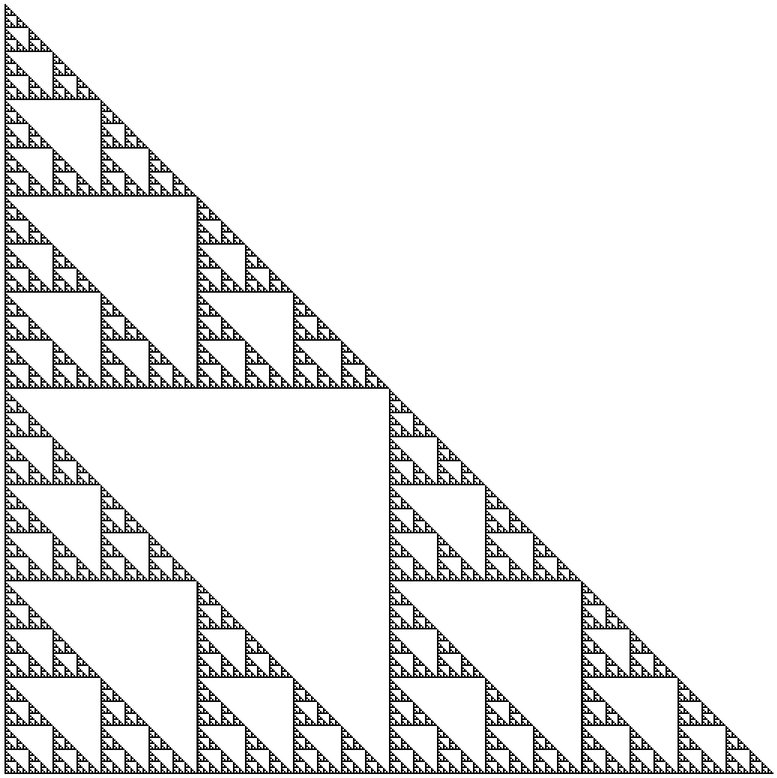

Fig. 3. A skew Sierpinski triangle.

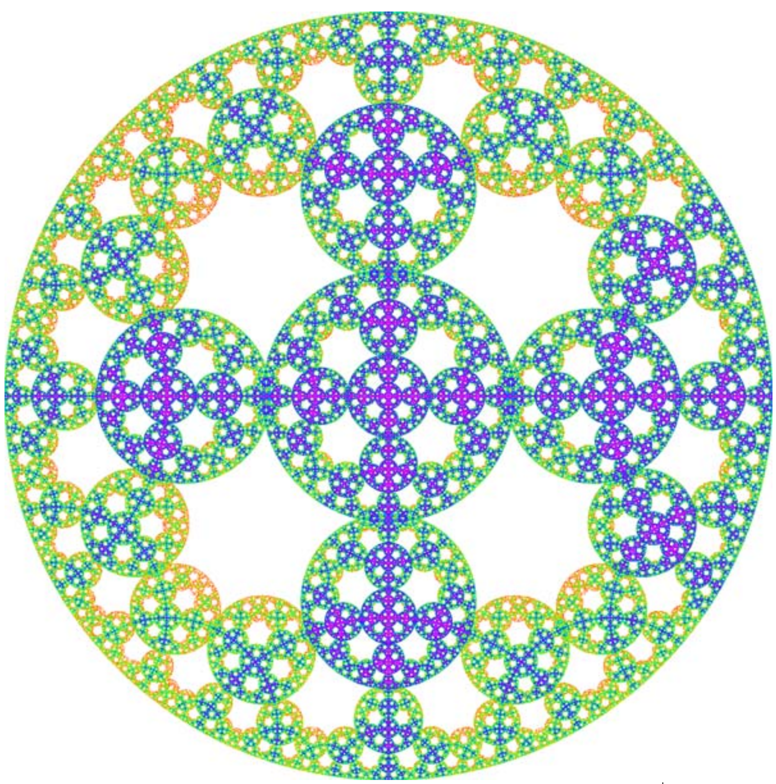

Fig. 4. An attractor with the symmetry of group $[4^+, 6]$.

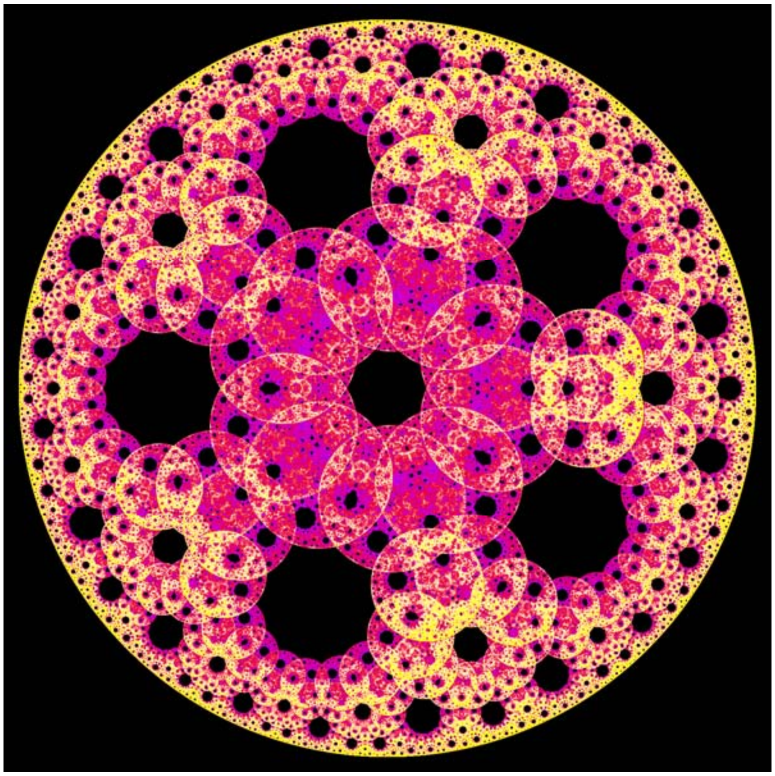

Fig. 5. An attractor with the symmetry of group [5,6].

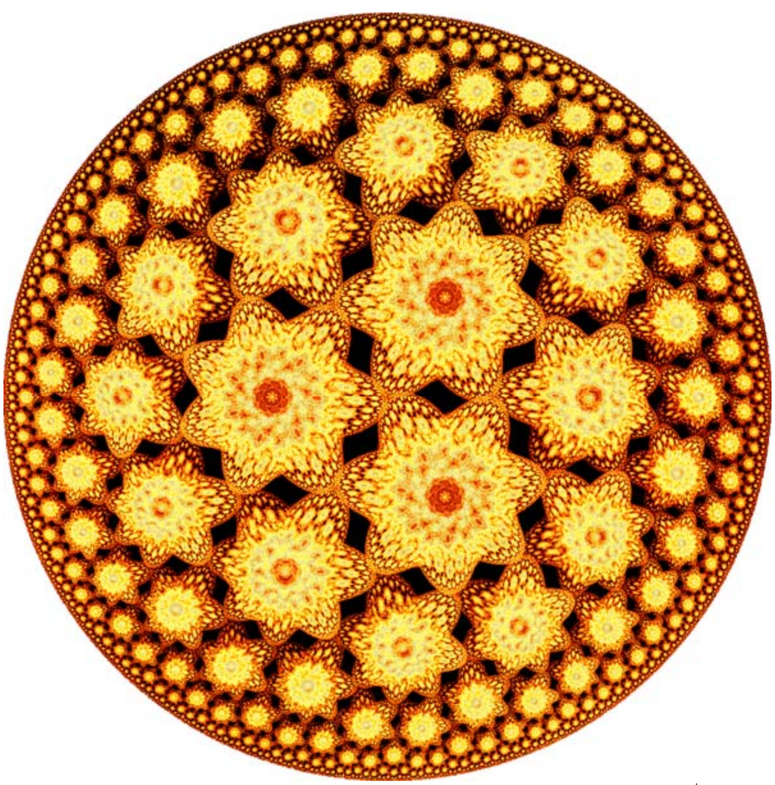

Fig. 6. An attractor with the symmetry of group $[3,7]^+$.

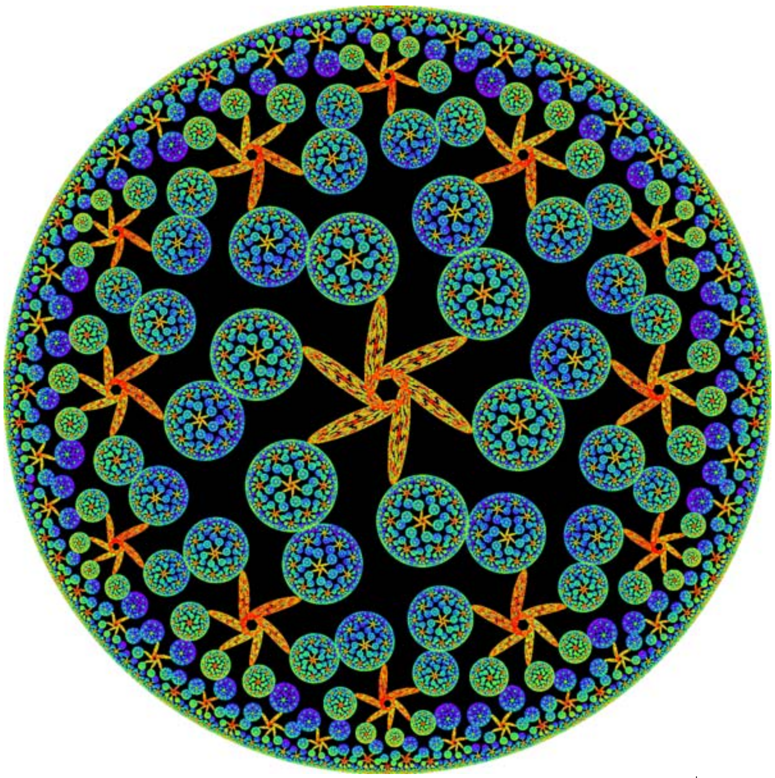

Fig. 7. An attractor with the symmetry of group $[6,4]^+$ created using two affine transformations.

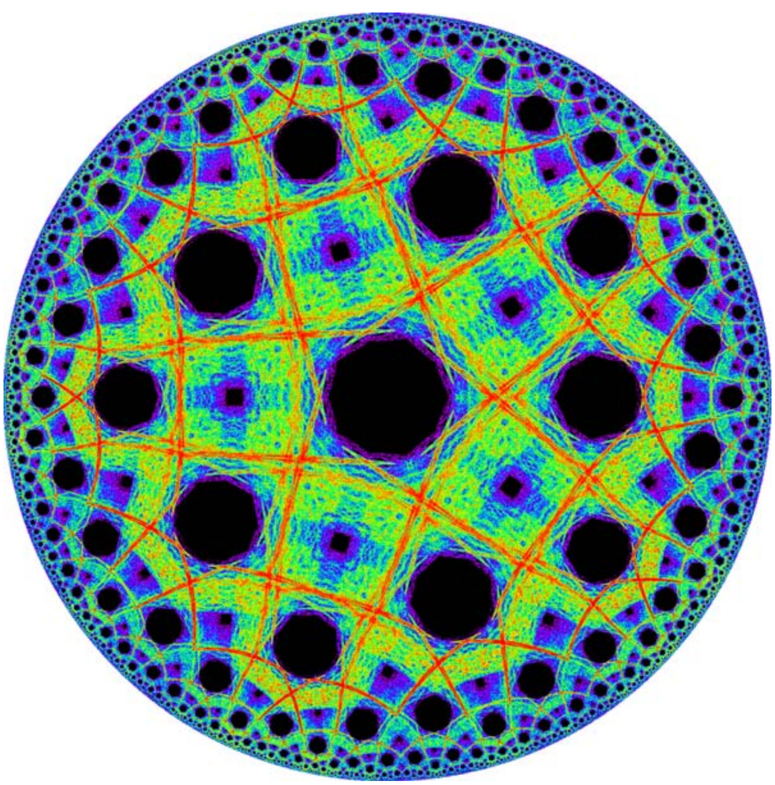

Fig. 8. An attractor with symmetry of the group [5,4] created using three random affine transformations.