Views of Fibonacci Dynamics

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Abstract
The Binet formula gives a natural way for Fibonacci numbers to be viewed as a function of a complex variable. We experimentally study the complex dynamics of the Fibonacci numbers viewed in that manner. Attracting and repelling fixed points are related to the filled Julia set and to regions of escape time images with fascinating behavior.

Introduction
The Fibonacci numbers are traditionally described as a sequence F_n defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. The sequence begins

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,....

The Fibonacci sequence has many remarkable properties, ranging from routine to startling [1-4]. Moreover, the numbers arise in nature, for example, as the number of spirals of pinecone petals. They may also be used to construct mathematical quasicrystals [5].

One of the beautiful formulas of Fibonacci numbers is the Binet formula. Binet described a version of the formula in 1843 [6-7]. Its beauty arises from the fact that the formula gives a closed form solution to a recursive definition, and from the symmetry of the formula itself. The Binet formula my be derived from the theory of difference equations, it can be derived by diagonalizing a suitable matrix, or it can be proven by induction [1-3]. The Fibonacci recursion has characteristic equation $x^2 - x - 1 = 0$ which has roots $\tau = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\bar{\tau} = \frac{1-\sqrt{5}}{2} \approx$ -0.618 where τ is the golden ratio and $\bar{\tau}$ is the conjugate of τ. Choosing constants to satisfy the initial conditions $F_0 = 0$ and $F_1 = 1$ gives the Binet formula: $F_n = \frac{\tau^n - \bar{\tau}^n}{\sqrt{5}}$. To obtain the Fibonacci numbers as a function of a complex variable, instead of viewing the index *n* in the Binet formula as an integer, we view it as a complex variable *z*. Thus we define the following complex Fibonacci function.

$$F(z) = \frac{\tau^z - \bar{\tau}^z}{\sqrt{5}}$$

The number $\bar{\tau}$ is negative and $\bar{\tau}$ appears as the base of an exponential in the Binet formula. Thus, complex numbers will result for fractional real arguments. Nonetheless, the Binet form gives a natural generalization of the Fibonacci sequence. It satisfies the initial conditions $F(0) = 0$ and $F(1) = 1$. It also satisfies the recursion $F(z) = F(z-1) + F(z-2)$ and it is defined for all complex values *z*.

Thus, we can ask questions about the complex dynamics of this function. What are its fixed points? Are they attracting or repelling? What happens upon iteration of the function? In this note we take a visual look at those questions and see that the Fibonacci numbers have interesting and beautiful complex dynamics.

Fixed Points

The fixed points of a function $F(z)$ are the values of z such that $F(z) = z$. Table I shows the values of the Fibonacci numbers at several integer values of z.

z	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$F(z)$	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21

Table I. Values of $F(z)$ at some integer points.

Notice that $z = 0,1,5$ are all fixed points. It might seem as though there ought to be another fixed point between -2 and -1 since $F(z)$ changes from negative to positive, but remember that since the definition of $F(z)$ involves an exponential with a negative base, we get complex values for $F(z)$ at intermediate values. For example, $F(-1.5) \approx 0.217287 - 0.920442i$. There appear to be many complex fixed points. For example, there is a fixed point near $-2.00376 - 0.197445i$.

The fixed points of $F(z)$ correspond to the zeros of $F(z) - z$. If we look at the magnitude of $F(z) - z$ along the real axis, we get the function shown in Figure 1. Note that the figure shows $z = 0,1,5$ are zeros and hence fixed points of $F(z)$ and that it appears that there are no other real fixed points.

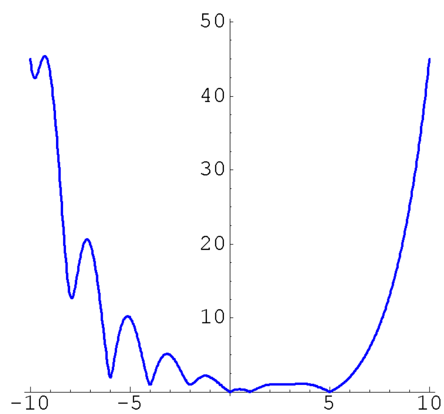

Figure 1. The magnitude of $F(z) - z$ along the real axis.

The situation off the real axis can be examined by looking at a false colored contour plot of the magnitude of $F(z) - z$. Figure 2 shows such a plot where $-36 \le \mathrm{Re}(z) \le 36$ and $-36 \le \mathrm{Im}(z) \le 36$. The lowest points are shown in black and higher points via hues running from red to magenta (highest). Notice there is a large black region near the center. There are some black regions appearing in a sequence above the center and others in a sequence mostly running to the upper left. This suggests that there are infinitely many fixed points in the complex plane in the upper left quadrant. Table II gives the values of $F(z)$, its derivative, and the magnitude of that, at the fixed points $z = 0,1,5$.

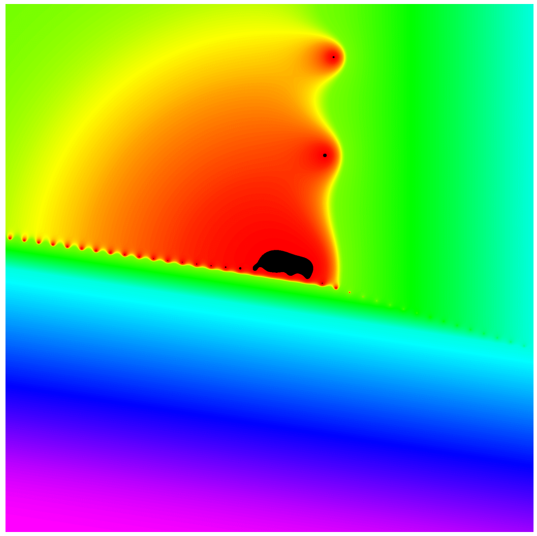

Figure 2. The magnitude of $F(z)-z$ in the complex plane for $-36 \le \mathrm{Re}(z), \mathrm{Im}(z) \le 36$.

z	$F'(z)$	$\lvert F'(z) \rvert$
0	0.430409-1.40496i	1.46941
1	0.215204+0.868315i	0.894586
5	2.36725+0.126685i	2.37064

Table II. Derivatives at Some Fixed Points.

The magnitude of the derivative at $z=0$ and $z=5$ is greater than 1. That implies those are repelling fixed points. However, the magnitude of the derivative at $z=1$ is less than 1, so this is an attracting fixed point. Thus, we expect some region around $z=1$ to not diverge to infinity, but instead, remain finite. The points in the complex plane that are eventually attracted to $z=1$ are called the basin of attraction of $z=1$. The set of points that do not diverge to infinity are the filled Julia set. When the Julia set is nontrivial, it has become common view such sets with an escape time image showing how quickly points outside the filled Julia set get large.

Escape Time

In particular, an escape time image corresponds to some region in the complex plane and typically color is used to indicate the number of iterations required before iterates get large. Perhaps the most famous illustrations of those occur for the famous quadratic Julia and Mandlebrot sets, but escape time images and basins of attraction have been utilized to visualize the dynamics of many processes [8-13].

In order to create an escape time image of a function $f(z)$, one uses an algorithm of the following type.

- Select a maximum iteration bound, *N*, and a sense of unbounded, *M*.
- For all pixels (j,k) corresponding to points *z* in a rectangular portion of the complex plane, do the following:
  - let $i=0$

• While $z < M$ and $i < N$ do
  • $z = f(z)$
  • $i = i + 1$
end while
• If $i = N$, mark the pixel (j, k) black, otherwise, mark the pixel a hue that corresponds to *i*.
end for all.

We apply this algorithm to $F(z)$ with $N = 512$ and $M = 10^{10}$. Figure 3 shows the escape time where $-6 \le \mathrm{Re}(z) \le 6$ and $-6 \le \mathrm{Im}(z) \le 6$. Red corresponds to rapid escape and other hues, running to magenta, correspond to slow escape. Notice the large black region on the right of center and many smaller regions. There are also fans of black regions, for example, a sequence of six of them appear to be marching across the red region in the lower half of that figure. A J [14] script that duplicates the image shown in Figure 3 is available at [15].

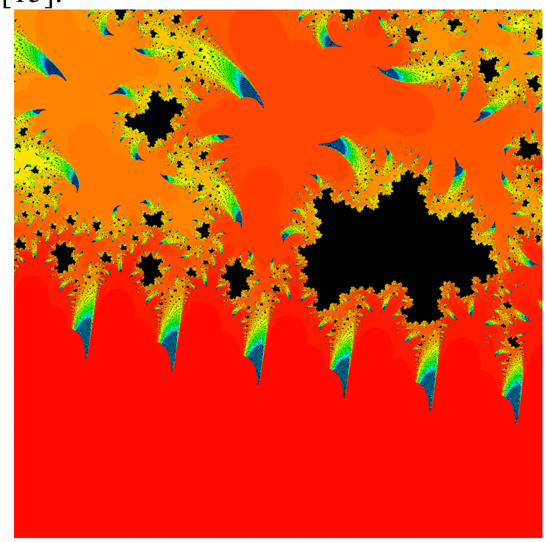

Figure 3. Escape time of $F(z)$ for $-6 \le \mathrm{Re}(z), \mathrm{Im}(z) \le 6$.

Figure 4 shows the escape time where $-36 \le \mathrm{Re}(z) \le 36$ and $-36 \le \mathrm{Im}(z) \le 36$. Notice the huge fans in a vertical sequence and the complex array of black regions in the upper left portion of the image. Figure 5 gives an image centered on the origin with width 1. Notice that there appears to be a spiral of fans, five fans per spiral, approaching the origin. An animation zooming toward the origin may be viewed at [15]. It reinforces that perception of the spiral. An animation zooming toward $z = 5$ may also be viewed at [15]; it shows that the repelling fixed point appears to be on the lower right edge of the large fractal black region that contains $z = 1$.

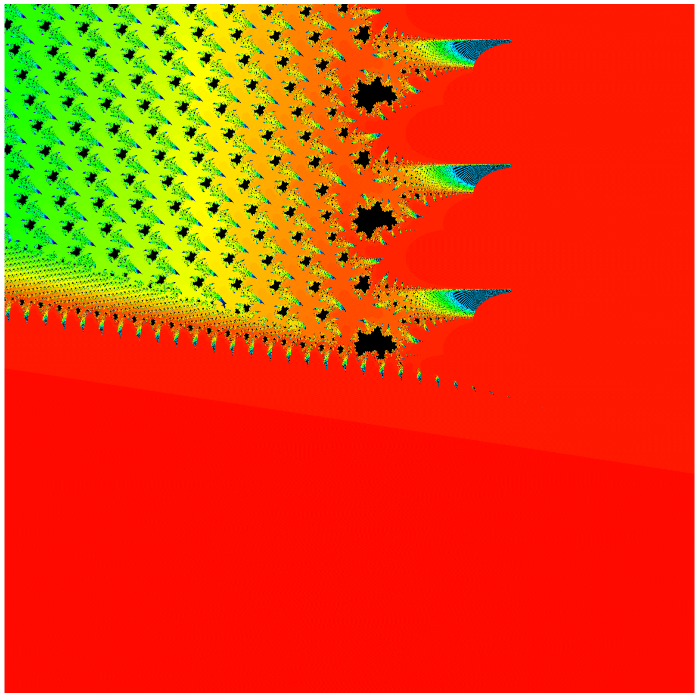

Figure 4. Escape time of $F(z)$ for $-36 \le \mathrm{Re}(z), \mathrm{Im}(z) \le 36$.

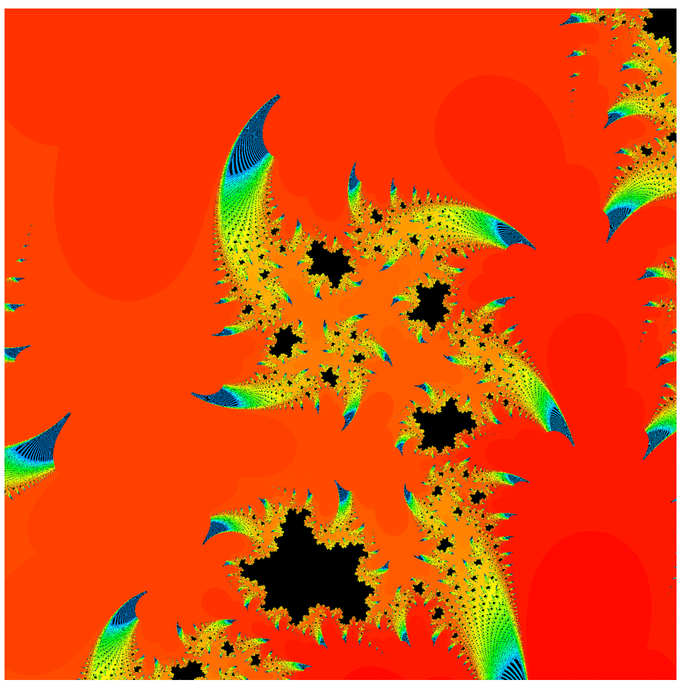

Figure 5. Escape time of $F(z)$ for $-0.5 \le \mathrm{Re}(z), \mathrm{Im}(z) \le 0.5$.

Figure 6 shows more detail of the large fan above and to the right of the origin. Notice the fan is a fractal array of fans and black regions. The Julia set for this function seems quite complicated.

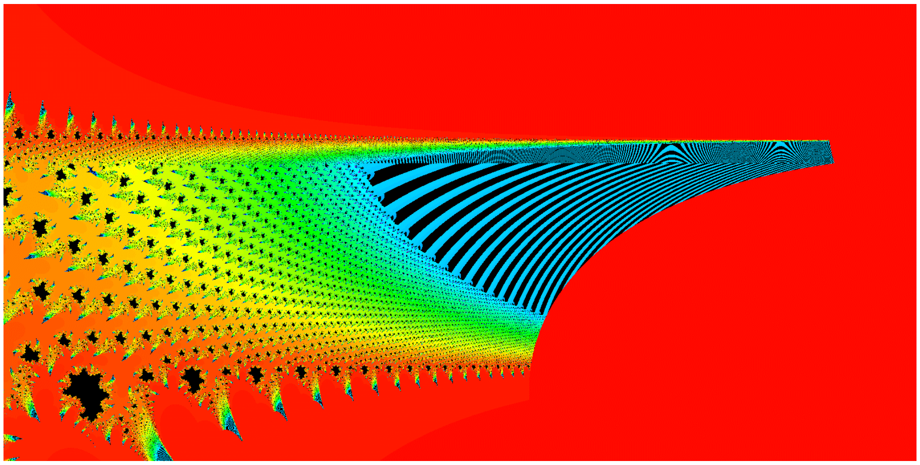

Figure 6. Escape time of $F(z)$ near $12+5i$.

Conclusions

By using the Binet formula we have been able to investigate the complex dynamics of the Fibonacci numbers. There are integer fixed points that are associated with a large basin of attraction, an edge of that basin, and a spiral of fans. There are additional complex fixed points, and the escape time images show the Fibonacci numbers have rich complex dynamics.

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