A Local Cellular Model for Snow Crystal Growth (preprint)

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Abstract

Snow crystals have a rich diversity of forms with striking hexagonal symmetry. The two-dimensional types include dendrite, stellar, sector and plate forms. Physical studies have shown that the particular form of a snow crystal is dependent upon the temperature and saturation in the growth environment. We investigate a simple, local, two-dimensional model that has just two parameters exhibiting the desired diversity. The dendrite growth is especially realistic. We use the model to investigate the impact of varying conditions of growth.

Introduction

Snow crystals exhibit remarkable, intriguing six-fold symmetry while displaying a wide diversity of forms. Kepler suggested in 1611 that the symmetry of snow crystals was related to the hexagonal packing of spheres [1], which is remarkable foresight given the modern understanding of the hexagonal molecular packing of ice crystals [2]. Figure 1 shows some of the diversity of naturally appearing snow crystals. These images are from the extensive collections of photographs of natural snowflakes by Bentley and Humphreys [3]. Nakaya made a wide-ranging scientific investigation of snow crystals, both natural and artificial, and also has provided extensive collections of photographs of snow crystals [4]. He found that the form of snow crystals depended heavily upon the

saturation and temperature during growth (he actually directly varied the air temperature and the temperature of a water bath used to introduce vapor). The forms observed include dendrites, stellar forms, sectors, plates, needles, columns, scrolls, and cups and many others. The snowflakes could combine features in endless variety; for example, dendrites extending from the corners of plates and plates appearing on the ends of long branches. He observed that many photographed images of snowflakes, especially those in [3], exhibit deterioration from sublimation and hence may not represent the most perfect snow crystals.

Our goal is to create a simple local model for snowflake growth that exhibits the variety of 2-dimensional forms seen in snow crystals. Motivated by Nakaya, we aim to use just two parameters. However, while we use general physical notions to design our algorithm, we are not trying to fit equations of the physics. The model we give is 2-dimensional, so we can not hope to model the forms of snow crystals that have substantial 3-dimensional components. For example, the needles and capped column shown in the lower right of Figure 1 are beyond the scope of our model. Nonetheless, we will see that we can easily exhibit simulations of the growth of crystals with sector, stellar, dendrite, and plate forms.

While all the forms of snow crystals have their beauty, the fernlike structure of dendrite growth is especially compelling. Dougherty, et al., have done extensive, more recent, work on dendrite growth in the laboratory [5-7]. Simulations based upon phase field models may be found at [8-9].

Wolfram has popularized a Boolean model for snowflake growth [10-11] that evolves on a hexagonal lattice by the following rule. At each time step, each cell is either

ice or not. On the subsequent step, cells that were ice, remain ice, while cells that were not ice, become ice if and only if exactly one of the neighboring cells is ice. While that model provides examples of abstract plates and sectors, it does not provide global dendrite or stellar growth. However, that model is offered in the spirit that Boolean hexagonal automata can have behavior surprisingly similar to a kind of snowflake growth. We present a far more realistic model, but work in a similar spirit since we sought a very simple local model.

Coxe and Reiter [12] used a local model with real values based upon linear combinations and counts of cells of ice appearing within a two level neighborhood. The coefficients, along with a background level, formed a set of five parameters used in that model. Sensitivity to background level appeared and a great variety of forms were easy to obtain, but the growth was rather abstract and dendrite growth, in particular, did not appear very physical. Additional early cellular growth models may also be found in the references to [12]. Also, like our model, a coupled map lattice is a local cellular model with real values in the cells. Such models for crystal growth based upon physics have been considered [13-14]. However, most of the work with coupled map lattices utilizes chaotic nonlinear maps for updates that are not physically motivated and which, of course, produce sensitive and complicated behavior. Another paper with good examples of simulated dendrite growth is [15]. Our model is simpler than those examples and varying the two fundamental parameters illustrates a broad range of growth patterns similar to those seen for natural and artificial snow crystals.

Model and Parameter Diagram

The model we investigate uses a hexagonal arrangement of cells and each cell contains a

real value. Very informally, we view the value of any cell as measuring the amount of

water at that cellular location. Values of one or higher are taken to correspond to ice,

while lower values are taken to represent water in a form that may possibly move to

neighboring cells. The automata we use can be described in two stages. Each stage may

be computed by using a nearest neighbor hexagonal real-valued cellular automaton, and

hence the algorithm we use could be viewed as a one-stage automaton on neighborhoods

containing two levels. The first stage is to determine the receptive sites. These are the

sites that are ice or that have an immediate neighbor that is ice. At the next stage, the

values of the cells are given by the values at the receptive sites plus a constant plus a

diffusion term. The diffusion term is a local average of a modified cellular field obtained

by setting the receptive sites to zero. The averaging will be discussed below. Figure 2

shows a diagram illustrating the computation for one small patch. Note that the zeros in

the array of unreceptive sites are used as values in the averaging while the zeros in the

array of receptive sites are placeholders. The motivation for the model is that receptive

sites are viewed as permanently storing any mass that arrives at that point. The mass in

the unreceptive sites is free to move, and hence moves toward an average value. Lastly,

the constant added to receptive sites corresponds to the idea that not only is "add a

constant" an especially simple generalization to the model, but it very informally captures

the idea that some water may be available from outside the plane of growth.

The constant to be added to the receptive sites is one of the parameters that we

vary. The second parameter that we vary is the background level. We will ordinarily

begin with a single cell of value one (an ice seed) in a sea of a constant background. We

will usually denote the added constant by γ and the background level by β.

The boundary conditions are taken to be fixed at the background level at a fixed

(Euclidean) distance from the initial cell. These boundary conditions attempt to make the

boundary conditions as isotropic as possible. However, different points on the boundary

are a different number of hexagonal steps from the center, so information takes different

time to flow from the center to different points on the boundary. Various weighted

averages may be used on the unreceptive field. We will use the average of the current cell

value with the average of its nearest neighbors. Thus, the center cell has weight $1/2$ while

the six neighboring cells each have weight $1/12$.

Figure 3 shows a diagram of the configurations resulting from several background

levels $0 \le \beta \le 0.95$ and addition constants $0 \le \gamma \le 1$. We embedded a single isolated seed

in an approximately 400 by 400 hexagonal arrangement of the background level. We

allowed the configuration to evolve until ice approaches the (circular) boundary or until

10,000 iterations had occurred. Notice the appearance of dendrites, stellar forms, sectors,

and plates. The darkest half of the gray values used in the image correspond to cell values

below 1, with black corresponding to 0. The lightest half correspond to ice; however, we

reverse the values in this case so that white corresponds to 1. Thus we get contrast at the

boundary between ice and not.

When $\beta = 0$ and $\gamma = 0$, no growth appears. When $\beta = 0$ and $\gamma > 0$, plates form,

the rate depending upon γ. Regardless of β, at high levels of γ, we get plates, sectors

or broad stellar forms. For low levels of γ and at low levels of β, we get dendrite

growth; for high levels of β we obtain stellar or more abstract forms. There is considerable variety of forms and the dendrites appear remarkably natural.

Figure 4 shows more detailed versions of selected choices of parameters illustrating thin dendrite growth, thick dendrite growth, dendrite growth that has nearly collapsed into a stellar form and a somewhat abstract sector form. On the thin dendrite one can observe many lengths of secondary dendrites and tertiary dendrite growth is visible. Notice the fairly regular pattern of secondary dendrite growth on the thicker dendrite. The lower left is the result of very regular dendrite growth where the growth overlaps and forms a kind of stellar form. This would be more apparent if a small amount of melting or blurring of the form occurred. The final example has a kind of sector form. We describe such growth as abstract since there are somewhat unnatural small gaps in the form. The Packard-Wolfram model has very similar features. Again, a small amount of blurring could lead to realistic forms for snowflakes.

Illustrations for additional values of β and γ, some different growth field radii, and some animations of the growth for selected parameters may be found at [16]. The animations there make it clear that the dendrite growth develops through phases. We also see the diversity of forms and basic qualitative behavior is maintained over the given range of radii.

Variation on Diffusion

The diffusion of the vapor field of the unreceptive sites at a position P and time t, denoted $u(t, P)$, is expected to satisfy the diffusion equation $\dfrac{\partial u}{\partial t} = a \nabla^2 u$ where a is the diffusion

constant and $\nabla^2 u = \dfrac{\partial^2 u}{\partial^2 x} + \dfrac{\partial^2 u}{\partial^2 y}$ is the Laplacian. The Laplacian may be approximated on

the hexagonal lattice via

$$\nabla^2 u \approx \frac{2}{3}\left(-6u(t,P) + \sum_{N \in nn(P)} u(t,N) \right)$$

where $nn(P)$ denotes the set of nearest neighbors of P. Thus, solutions may be

approximated on a hexagonal lattice by

$$u(t+1,P) \approx u(t,P) + \frac{\alpha}{12}\left(-6u(t,P) + \sum_{N \in nn(P)} u(t,N) \right)$$

where we have reorganized constants via $\alpha = 8a$ so that our model with center weight

1/2 and weight 1/12 for neighbors corresponds to taking $\alpha = 1$.

The actual growth is very sensitive to α. In our experience, when not taken to

extremes, the qualitative behavior of the (β, γ) diagram does not depend on the choice of

α. However, the details vary considerably. Figure 5 shows examples where the diffusion

constant is varied for $\beta = 0.4$ and $\gamma = 0.0001$. Notice that $\alpha = 0.5, 0.501, 0.502$ each give

very different pictures; however, they all exhibit dendrite growth. The figure also shows

the standard model when $\alpha = 1$ which appears very similar to the $\alpha = 0.5$ case. A pretty

growth pattern with many levels of dendrites occurs when $\alpha = 2.003$ and an extreme

situation, with explosive behavior, occurs when $\alpha = 2.69$. When α has reached that last

level, the original site is weighted a negative amount; hence the unnatural behavior is not

surprising. Animations showing the sensitivity of growth to α appear at [16].

Variations on Conditions

One of the advantages of having a simple model is the ability to study variations. We first consider modified initial conditions. Figure 6 shows some examples where small random configurations on patches of size $\rho = 3$ or $\rho = 5$ are used. The first shows dendrite growth with six different branches, but the basic type of growth seems similar on each branch. The second is similar, but with thicker growth. The $\rho = 5$ examples show more substantial changes from six-fold symmetry. They seem to have close to bilateral symmetry in two directions. The first is somewhat suggestive of the irregular form at the lower left of Figure 1. The last example has, striking gaps, and clearly visible fourth level dendrite growth. A $\beta - \gamma$ table giving some additional examples for $\rho = 3$ can be found at [16].

Secondly, we consider some examples of what happens when the growth parameters are changed partway through growth. This is motivated by the idea that naturally occurring snow crystals form as they fall through air with different temperature and humidity. Figure 7 gives two examples. The first begins with parameters giving plate growth, then γ is changed so that dendrite growth occurs. The second begins with dendrite growth, the unreceptive field is then proportionally changed to meet new boundary conditions, and growth continued with parameters that correspond to abstract sector/plate growth. We see it is fairly straightforward to create slightly imperfect crystal growth and to combine growth forms on the same crystal.

Conclusions

A simple model on a hexagonal array of cells shows many of the growth forms seen in naturally occurring snow crystals. The model uses real valued cells and a distinction between receptive and unreceptive sites is made using a threshold. Except for the threshold, the updates are smooth. Like the saturation and temperature controlled in laboratory snow crystal experiments, the model has just two parameters. Varying those parameters results in realistic dendrite growth, along with plates, sectors and stellar forms. We see that changing the initial seed configuration and changing the growth parameters can result in growth with realistic appearance. While this two-dimensional model can not model the forms of snow crystals that are distinctly three-dimensional, it does a remarkable job for the two-dimensional ones. We hope that this simple model inspires further investigations using discrete local cellular models for the study of phenomenon that exhibit sensitive behavior with a diversity of forms.

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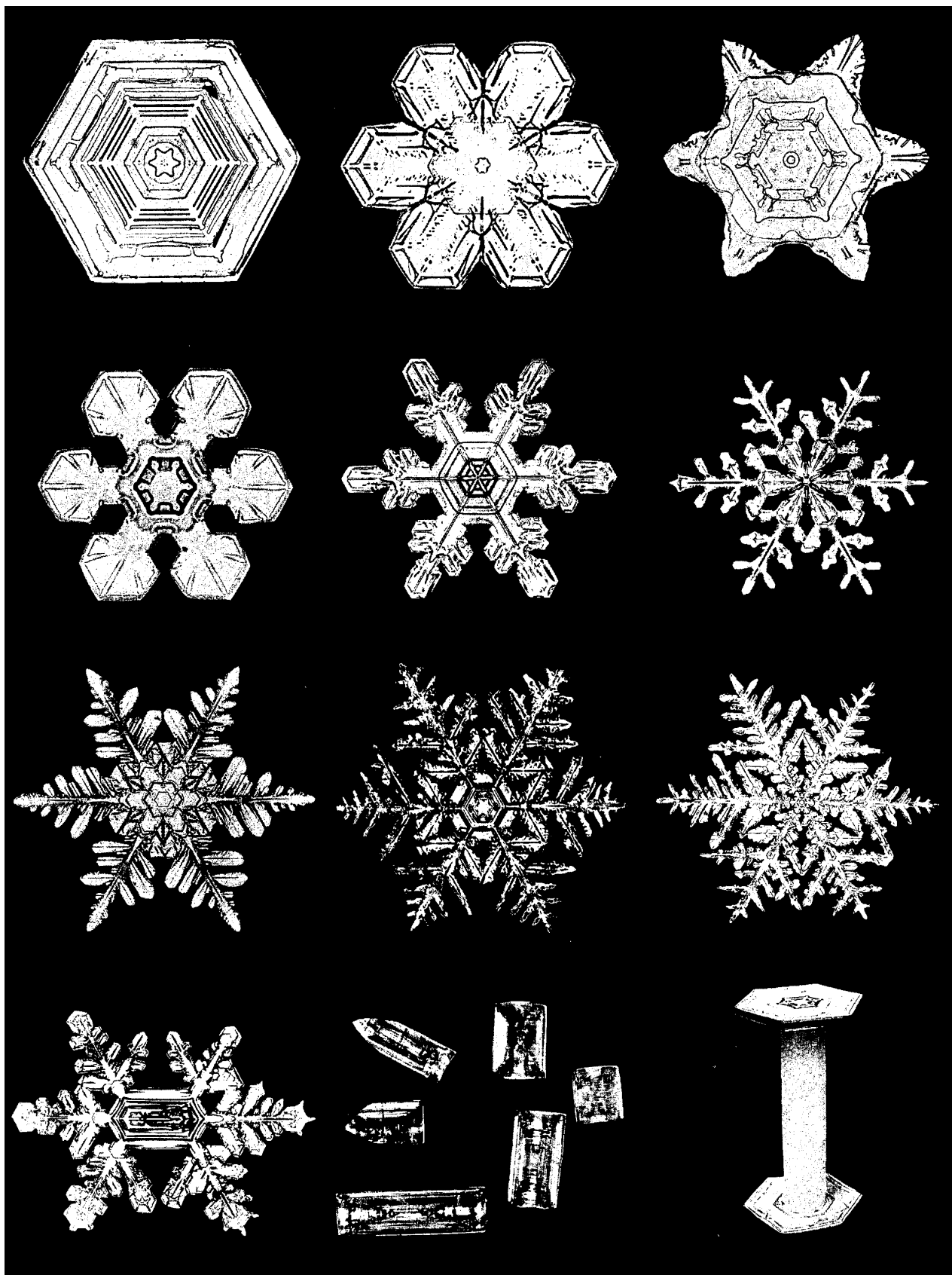
Figure 1. Snow crystals: plates, sectors, stellar, dendrite, irregular and three dimensional.
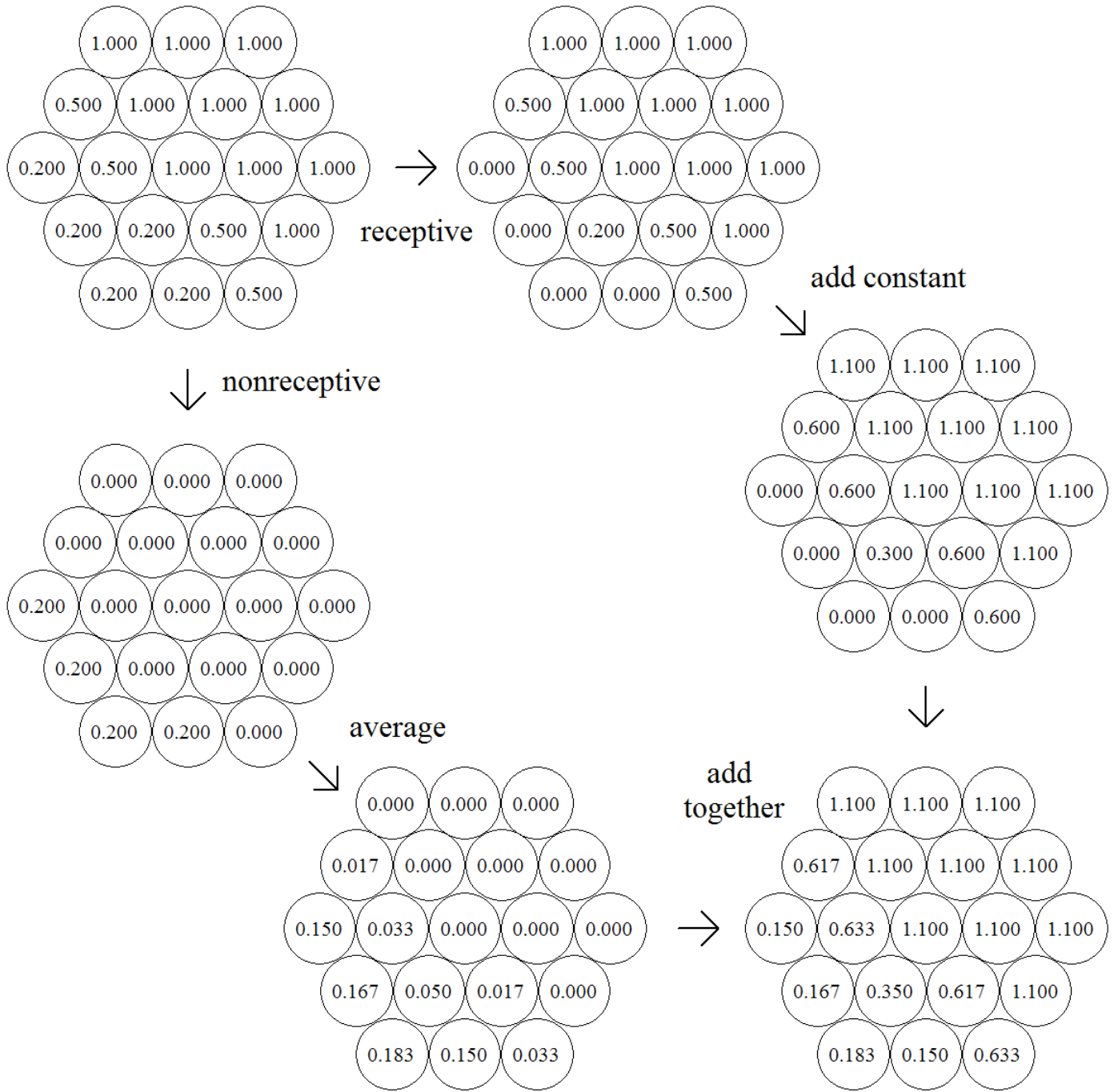
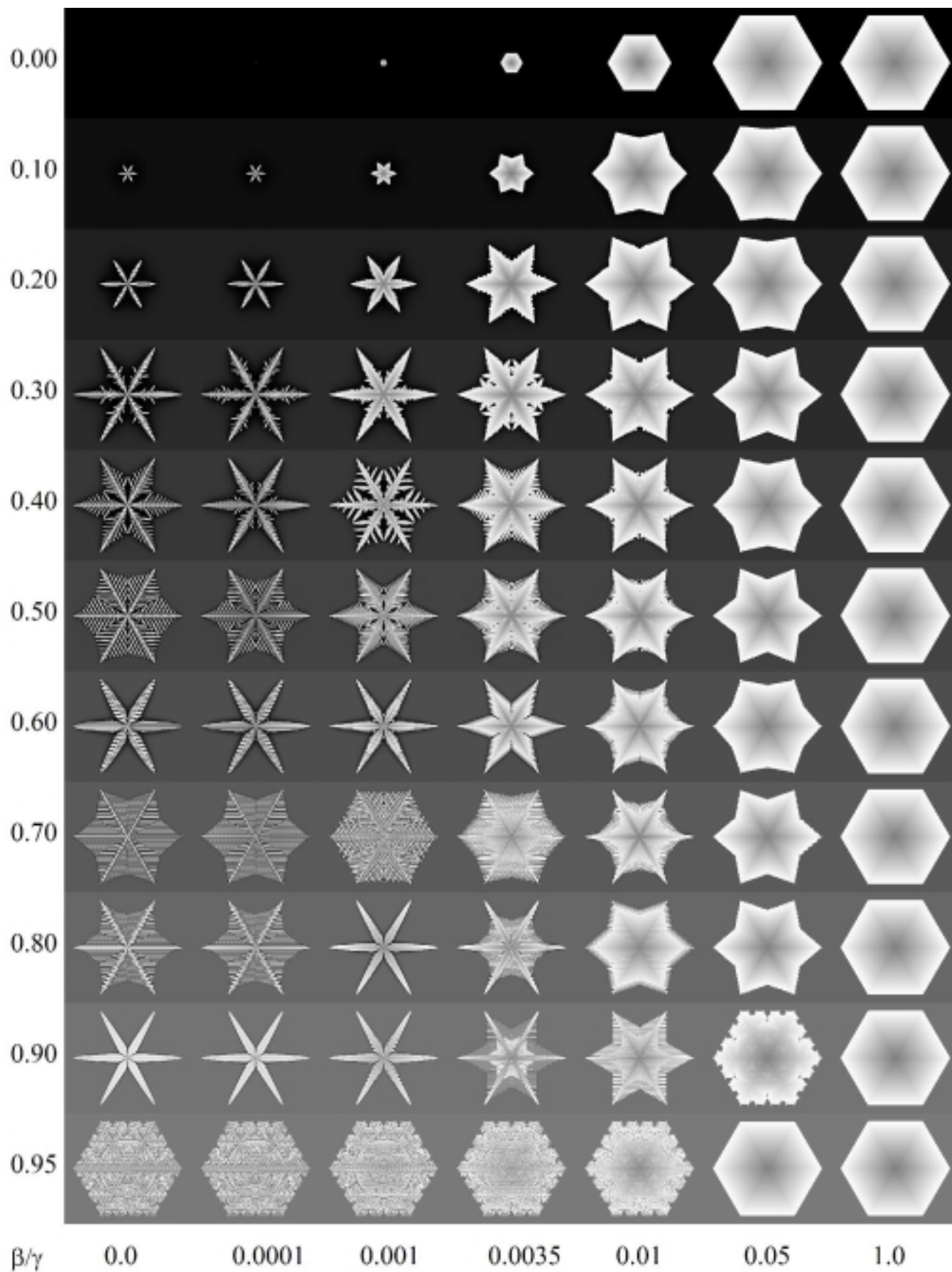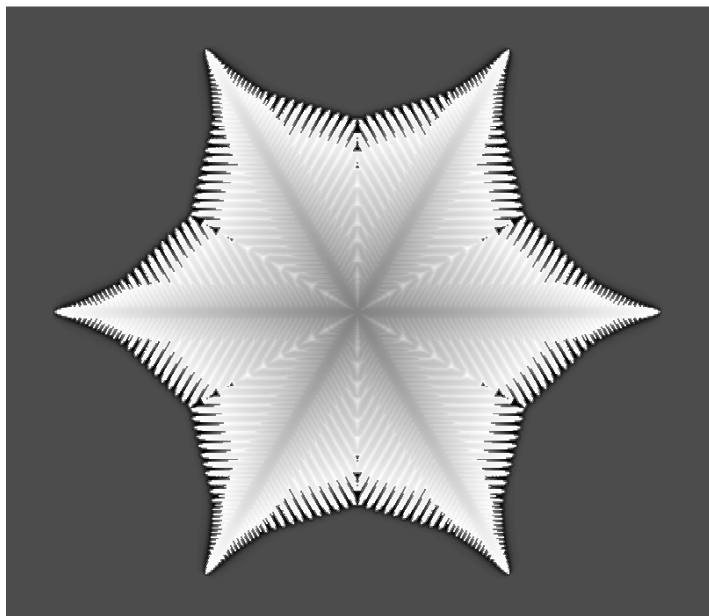Figure 2. Illustration of the algorithm on a small patch.

Figure 3. Growth forms that appear as β and γ vary.

β = 0.3, γ = 0.0001

β = 0.35, γ = 0.001

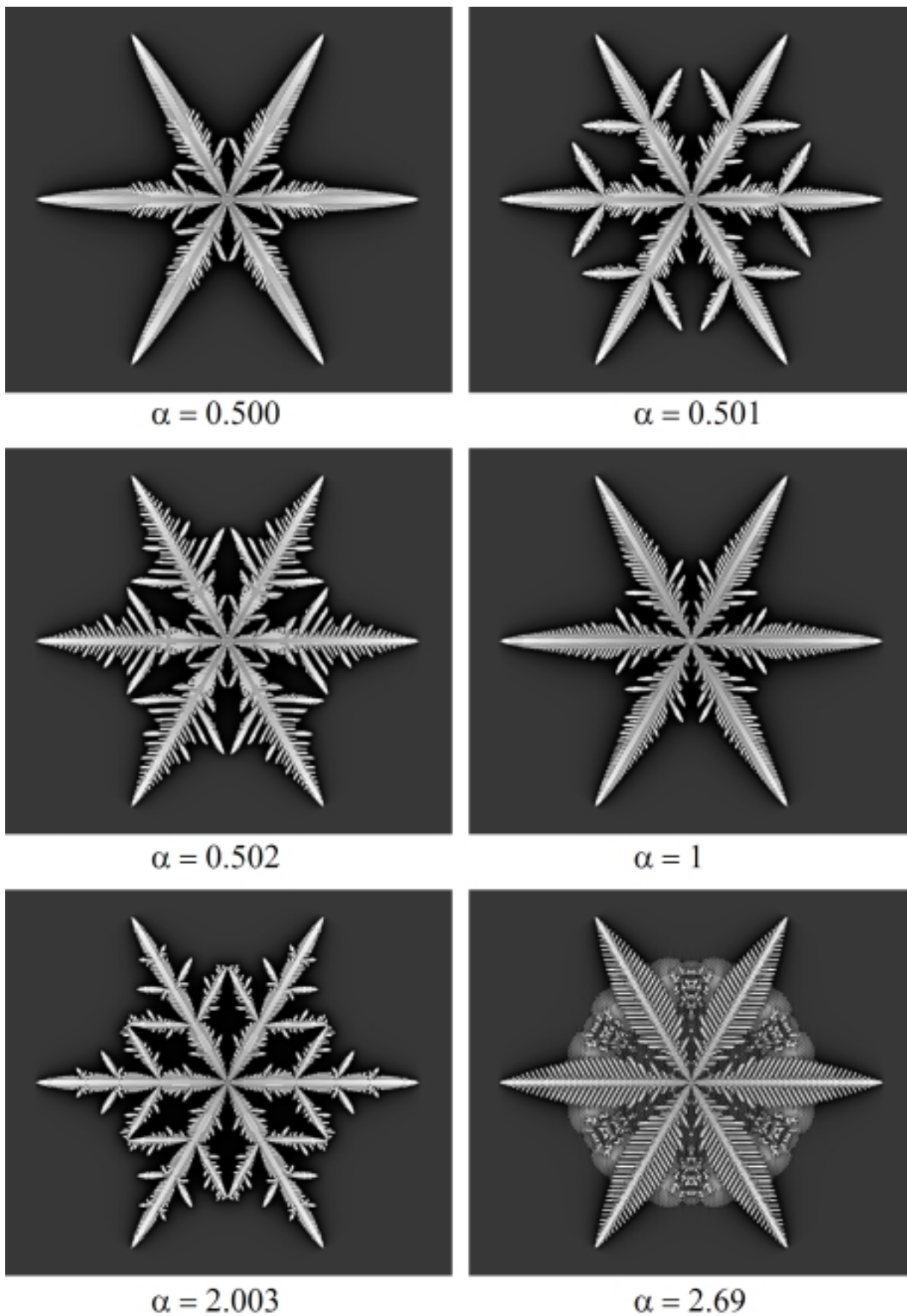β = 0.6, γ = 0.01
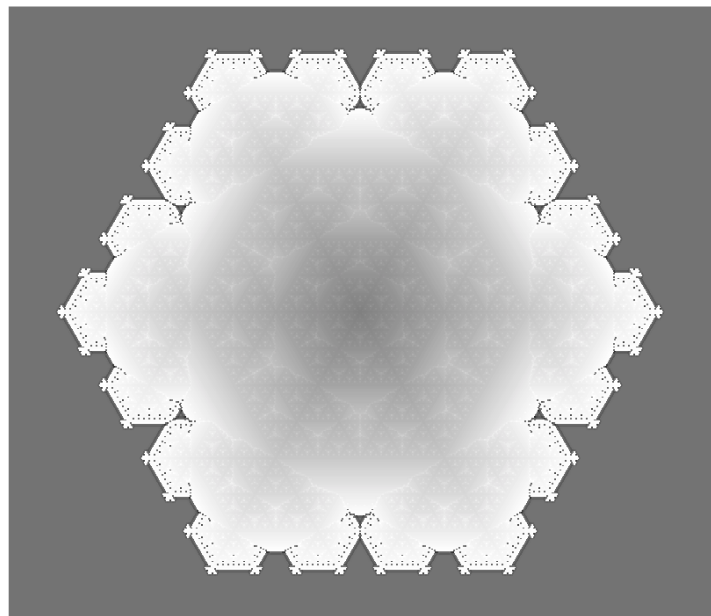
β = 0.9, γ = 0.05

Figure 4. Selected dendrite, stellar and sector forms.

Figure 5. Variety of forms appearing for different diffusion constants when $\beta = 0.4$, $\gamma = 0.0001$

ρ = 3, β = 0.35, γ = 0

ρ=3, β = 0.4, γ = 0.001

ρ=5, β = 0.4, γ = 0.001

ρ=5, β = 0.55, γ = 0.001

Figure 6. Growth from a random initial cluster.
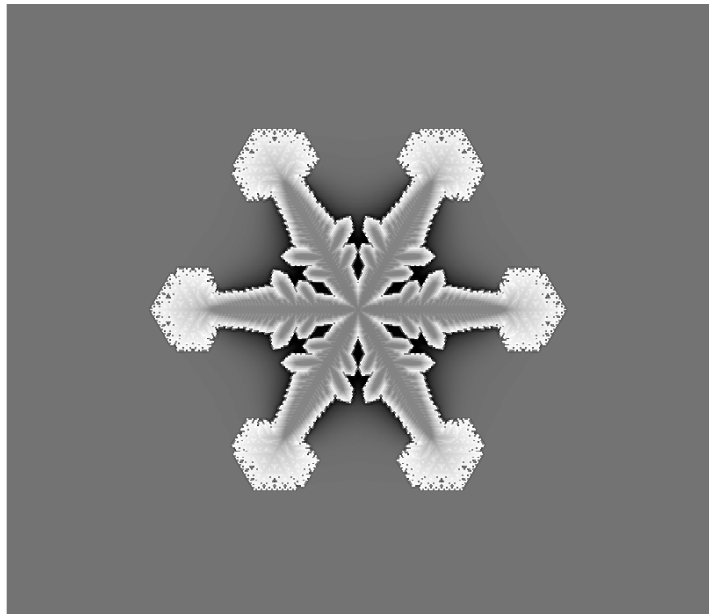
(β,γ) = (0.4, 0.5) & (0.4,0.001)   (β,γ) = (0.35, 0.0001) & (0.9,0.08)

Figure 7. Growth when parameters are changed.