THE GAME OF LIFE ON A HYPERBOLIC DOMAIN

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Abstract -- Classic two dimensional outer totalistic automata such as Conway's Game of Life can be generalized to any domain of cells where each cell has eight neighbors. A circular hyperbolic arrangement of cells is described where each cell has eight neighbors and the time evolution of these hyperbolic automata are visualized in three dimensions. Both random and symmetric initial conditions are investigated using these techniques leading to insight into typical long term behavior and examples of symmetry breaking.

1. INTRODUCTION

Cellular automata in two dimensions are typically defined on an infinite rectangular array of cells. The essential features of an automaton is that at each generation the state of each cell is updated using uniform local rules and the allowed values of the states of cells are from some finite set. Perhaps the most famous cellular automata is Conway's Game of Life [1, 2, 3]. This is a two state automata with each cell alive or dead at each state according to some simple rules. Screen savers and shareware programs running Conway's Game of Life as an animation are common. Observers quickly see that very complicated behaviors may occur but that the states tend to settle into a simple behavior after some iterations. One also notices that the long term behaviors observed often have some local square symmetry that is presumably an artifact of the rectangular arrangement of the cells and the symmetric rules.

More generally, automata are used in diverse fields such as image processing and for modeling a wide variety of behaviors ranging from the growth of a snowflake to the flow of a fluid [4-6] though these models may use very large numbers of states. Often these automata arise as discrete approximations to the solution of continuous problems. A rectangular arrangement of cells is usually the most convenient and simplest arrangement to use. As we have noted, this leads to artifacts with square symmetry being common. Other arrangements of cells would be expected to have their own artifacts, but would be more effective for certain models. In two dimensions, triangular and hexagonal domains have been used [4-8]. In particular, hexagonal lattices have been seen to be effective for gas automata since square lattice gas automata conserve rotational symmetries seen in physical gases [7, 9]. In three dimensions, the Game of Life has several proposed analogues [10-11] and three dimensional automata have also been studied on nonrectangular lattices [4, 12].

In this note we design a circular hyperbolic array of cells that shares with rectangular designs the fact that each cell has eight neighbors. As we move outward, the number of cells in each ring doubles and hence the number of neighbors within a fixed distance of a cell has the hyperbolic quality of being exponential rather than quadratic. One of the advantages of a circular domain is that we will see symmetry breaking. A classic physical example of this behavior is Harold Edgerton's “an instantaneous photograph of a ‘splash’ of milk” that appears in Thompson's “Growth and Form” [13] and also in [14]. This photograph shows how a droplet of milk with full circular symmetry breaks up into a discrete circular pattern with 24-fold symmetry; Thompson [13] also contains another photograph showing the situation after the splash subsides and circular symmetry is restored. In our case, we will see initial configurations with cyclic symmetry can break into configurations without symmetry followed by some symmetry restoration.

2. STABILITY AND HYPERBOLIC LIFE

The rules of the Game of Life are fairly simple: a cell that is alive at one generation will be alive at the next generation if 2 or 3 of its eight neighbors are alive; a cell that is dead at one generation will become alive if it has exactly 3 of its eight neighbors alive; all other cells will be dead at the next generation. This automata is well defined so long as each cell has eight neighbors. If we imagine concentric rings of cells where there is a doubling of the number of cells at each level, then each cell naturally has eight neighbors: two at the same level, two at the previous level and four at the next level. Figure 1 shows a configuration of cells with that feature. Notice that the center needs to be handled carefully: if there were a single central cell it would only have two neighbors. Choosing five central cells causes the central cells to also have exactly eight neighbors though the edge topology is different for center cells and cells at higher levels. Nonetheless, all cells have eight neighbors and we can uniformly apply the rules of the Game of Life.

In order to show cells as circles we use the arrangement of cells shown in Figure 2. This organization highlights the hyperbolic arrangement of cells. We next consider several configurations which are stable in hyperbolic life. Figure 3 shows four examples. First notice that a complete "ring" isolated from interference is stable since each cell in the ring has 2 neighbors. A two by two block, which we will call "footprints", as seen on the upper left is stable since each cell has 3 neighbors. The configuration of cells on the upper right, a "paw", is also stable since each cells has 2 neighbors. Lastly, an "arc" consists of a partial ring with cells hanging off of the end toward the center. Notice that if there were not cells hanging off the ends of the partial ring, then at each generation the partial ring would diminish by one cell on each end. In practice these arcs appear, but often the cell at the end is replaced by some more complex structure that yields some sort of "compound arc". Sometimes these compound arcs fold back on themselves giving a "loop".

In implementing the Game of Life on a rectangular array of cells the choice of boundary conditions plays an important role in behavior. While periodic boundary conditions are the least disturbing, it is best to increase the size of the array so that alive cells avoid the boundary. Programming this can be quite tricky as configurations know as gliders commonly arise from random configurations. These configurations replicate themselves slightly translated after 4 generations but this means the diameter of the region containing alive cells may grow without bound. In hyperbolic life, things at first glance are more frightening since the number of cells within a distance n of a fixed cell is exponential in n so that one might expect the size of the region containing alive cells to grow exponentially with the number of steps. Indeed, this would be true for general hyperbolic automata. However, notice that each cell outside the central ring has only two neighbors at lower levels. Thus, in hyperbolic life, where new life requires three neighbors, it is impossible for a cell become alive if it is outside of the levels currently containing alive cells. Thus, one never needs to expand the size of the domain used in hyperbolic life. This has the effect that hyperbolic life is simpler than the traditional Game of Life since the diameter can not grow without bound; however, hyperbolic life is remarkably interesting given its bounded behavior.

Figure 4 shows the result of the time evolution of hyperbolic life on a random initial configuration. The time evolution proceeds in an upward direction and colors are used to indicate the number of neighbors each alive cell has according to a scheme that is close to the scheme used in [15] for the traditional Game of Life:

- 0 neighbors: gray
- 1 neighbor: red
- 2 neighbors: green
- 3 neighbors: blue

· 4 neighbors: cyan
· 5 neighbors: magenta
· 6-8 neighbors: yellow.

Thus, gray and red cells die, green and blue cells persist to the next generation and cyan, magenta and yellow cells die. Notice that the long term behavior in Figure 4 shows the center cells dying out rapidly, footprints appearing in the back left, an arc on the right, a compound arc on the front right and two loops. Figure 5 gives a second example of the result of a random initial configuration: a loop, footprints and a paw persist in the long term. Notice the center is active far longer in this example and that a partial ring dies out one cell at a time in the lower left resulting in a green triangle with red top. These examples give an indication of how hyperbolic life tends to reach a stable configuration quickly — faster than the traditional Game of Life and reminiscent of the three dimensional version of the Game of Life in [11]; nonetheless, nontrivial configurations often appear.

3. SYMMETRY BREAKING

One of our goals in investigating hyperbolic life was to observe symmetry breaking. When we describe the symmetry of a configuration we are only considering the arrangement of the alive cells. Thus we can have 8-fold rotational symmetry by leaving levels 0, 1 and 2 empty and making every fifth cell at level 3 alive. Since there are 40 cells at that level, this gives the desired symmetry. In general one can also observe reflective symmetries, but we will usually only consider *n*-fold rotational symmetries. For example, Figure 6 shows two iterations of a configuration that has 20-fold rotational symmetry and that results in a 2-cycle. Notice the inner cells are stable while the cells in the outer ring alternate between even and odd positions.

Figure 7 shows the first 11 iterations of a configuration that begins with 4-fold rotational symmetry. Successive iterations are laid out like English text: across rows. At iteration 2 the symmetry has broken to 2-fold and at iteration 4 there is no rotational symmetry (keep in mind the innermost ring has 5 cells). At iteration 5 the 2-fold symmetry is recovered and at iteration 6 part of the configuration begins a 4-cycle though the four "footprints" don't stabilize until iteration 8. Figure 8 shows 24 stages of the time evolution of the configuration from Figure 7. Notice the footprints, the 4-cycle and the 2-fold symmetry.

Figure 9 shows the first 11 iterations of a configuration that begins with 8-fold symmetry. At iteration 2 this breaks to 4-fold symmetry, at iteration 3 it is 2-fold and there is no rotational symmetry in iteration 4. However, at iteration 5 there is 4-fold symmetry. The symmetry then breaks, first to 2-fold symmetry and then disappearing. Notice that while there is no rotational symmetry in the long term, there is bilateral symmetry. Figure 10 shows the time evolution of this configuration. Notice that in the long term it results in a 3-cycle with no rotational symmetry though the bilateral symmetry persists.

4. COMPUTATIONS, OBSERVATIONS AND CONJECTURES

We implemented hyperbolic life in the programming language J [16] and we piped the output, in the style of [17], to the raytracer POVRAY [18] to create the color images. J is a high level dialect of APL that is functional and ASCII based; nonetheless, like APL code, J code tends to be unreadable to readers unfamiliar with the language. On the other hand, the language is fairly easy to learn and it provides an effective environment for exploring hyperbolic life. Interested readers are welcome to contact the author for J scripts implementing hyperbolic life.

We sought examples of symmetry breaking via Monte Carlo explorations of the space of initial configurations with specified symmetry and specified size. Our explorations tracked the symmetry breaking, which was too common to be interesting, and also sought out periodic long term behavior as seen in the examples in the previous section.

Our experience with these explorations suggests the following observations and conjectures. We never observed the breaking of 5-fold symmetry. Indeed this must be the case because of the symmetry of the rules and the overall 5-fold symmetry of the domain. That is, since we can always make a fifth of a rotation a configuration with 5-fold symmetry must maintain that symmetry in the next generation. We also only observed configurations with 2^n-fold symmetry break by one factor of 2 yielding 2^{n-1}-fold symmetry. It is also the case that this is true in general. First notice that the only rotational symmetries possible are factors of $2^n \cdot 5$. If the factor of 5 appears, then this is equivalent to having both 5-fold symmetry and 2^k-fold symmetry. We have already seen that the 5-fold symmetry is preserved. Now suppose that the 2^k-fold symmetry occurs on a configuration with alive cells on levels j and higher. Any cells at the next generation at level j or higher must have the 2^k-fold symmetry. If 2^k divides the number of cells at level j-1, that is, if $k \leq j-1$, then any new cells at level j-1 also have the 2^k-fold symmetry. If not, then $k = j$ and the arrangement of cells at level j-1 have 2^{k-1}-fold symmetry. Since the original configuration can also be viewed as having 2^{k-1}-fold symmetry, so must the next generation. In summary, the only way for symmetry to break is by a factor of 2 forced by the lack of sufficient symmetry at a lower level where alive cells are appearing. As we saw in Figure 9, it is possible for the symmetry to increase by more than a factor of two. In our Monte Carlo investigations we never observed the symmetry rising from none to 5-fold symmetry. However, we see that we can pick a stable configuration with 5-fold symmetry at high levels and put four cells at level 0. One iteration would result in changing from no symmetry to 5-fold symmetry. A similar argument shows that any possible rotational symmetry can arise from a configuration without rotational symmetry.

In our investigations we observed configurations containing n-cycles where n was 1, 2, ..., 8, and 13. We see no reason that there are any restrictions on the period sizes; hence we conjecture that there are configurations of hyperbolic life that give rise to cycles of each length. We have seen that the active levels of hyperbolic life can not expand and hence there is no hope for a “glider” moving off toward infinity. Thus, like the hexagonal (3422) rule in [7] we see rich stable structure without apparent gliders. However, it is not clear whether there could be analogs of gliders that would move about in a circle. If we imagine a cluster of cells evolving counterclockwise it will need to pass through the 5 rays emanating from the origin; refer to Figure 1. It is possible to imagine a sequence of steps that might light each cell on the counterclockwise side of a ray. However, none of the cells further counterclockwise would have 3 neighbors. Hence there is a barrier if the hyperbolic life glider is to avoid the origin. Thus, a hyperbolic life glider would need to be very active near the origin and pass life past the barrier from level 0. It would be better to describe such a configuration as a spinning arm, rather than a glider. We have found no such configurations, but we conjecture that these spinners exist.

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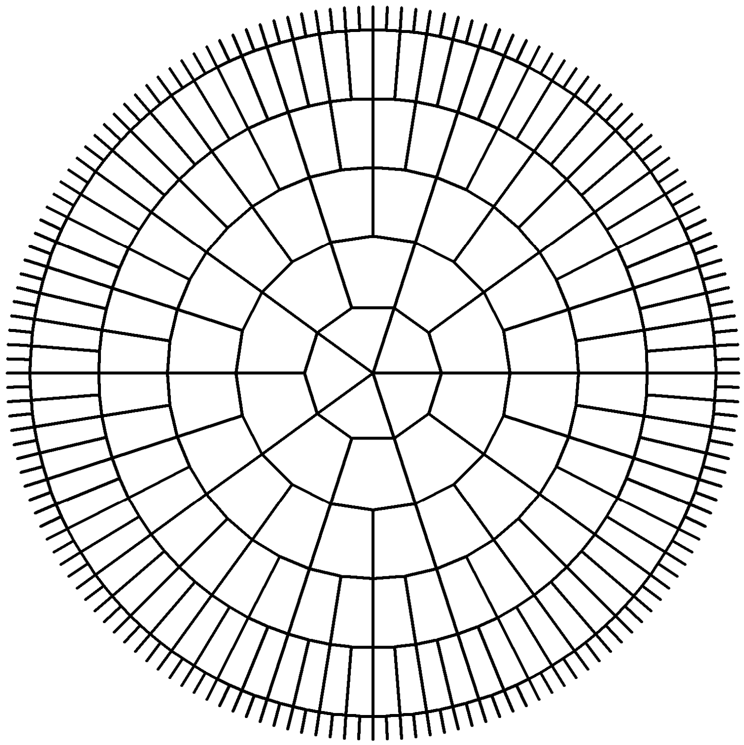

Figure 1. Circular arrangement of cells with each cell having eight neighbors.

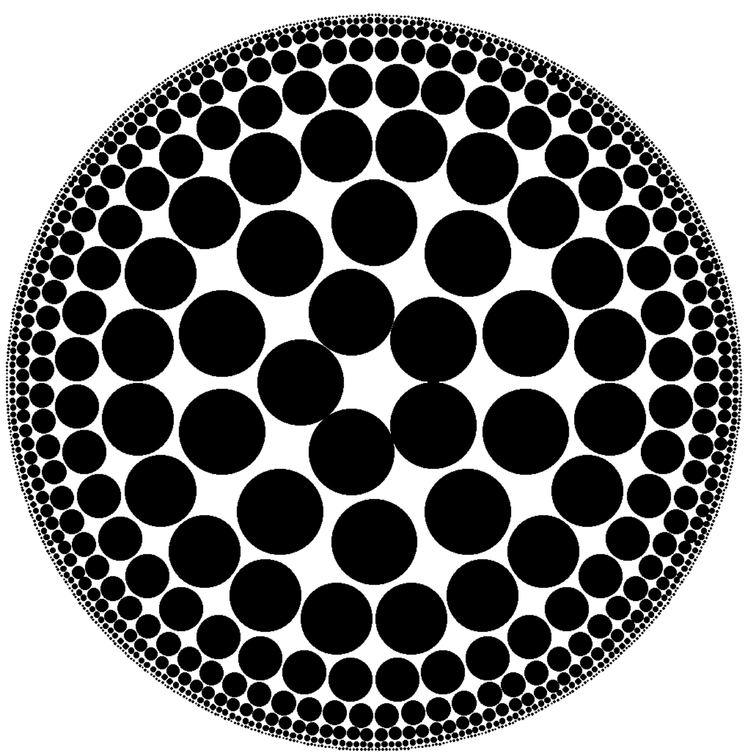

Figure 2. Hyperbolic arrangement of circular cells.

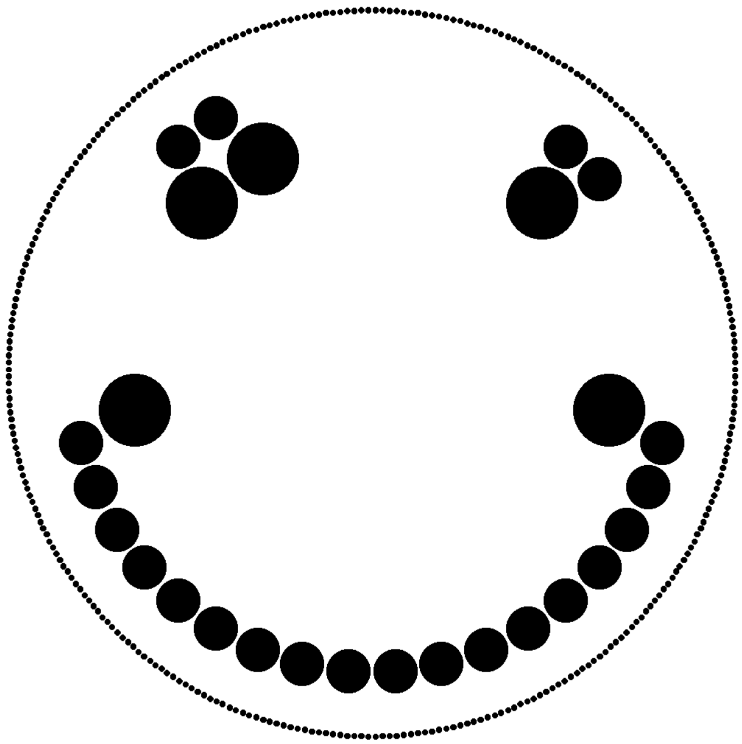

Figure 3. Some stable configurations for hyperbolic life.

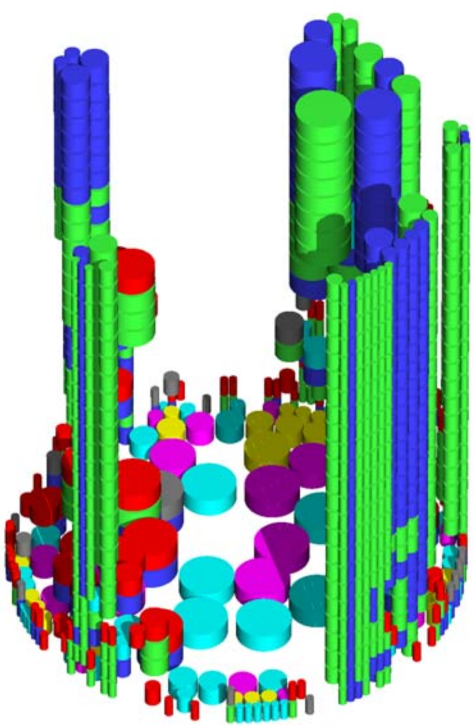

Figure 4. The time evolution of hyperbolic life on a first random initial configuration.

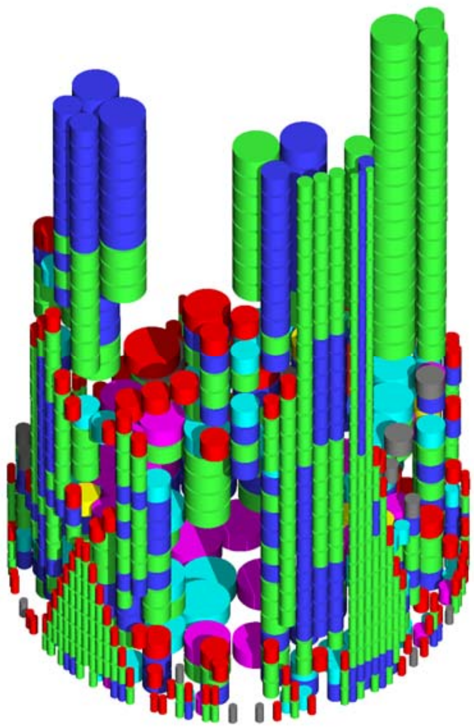

Figure 5. The time evolution of hyperbolic life on a second random initial configuration.

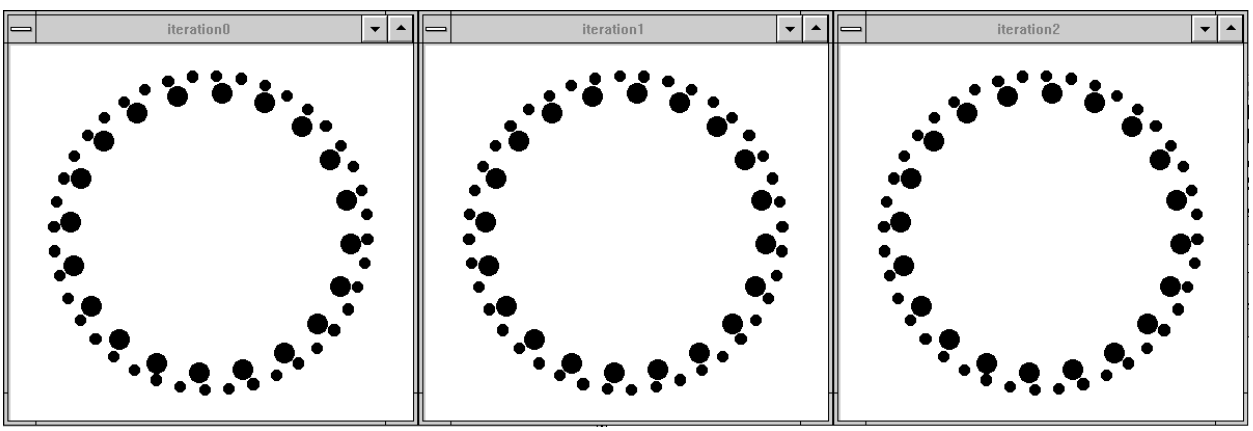

Figure 6. A 2-cycle with 20-fold rotational symmetry.

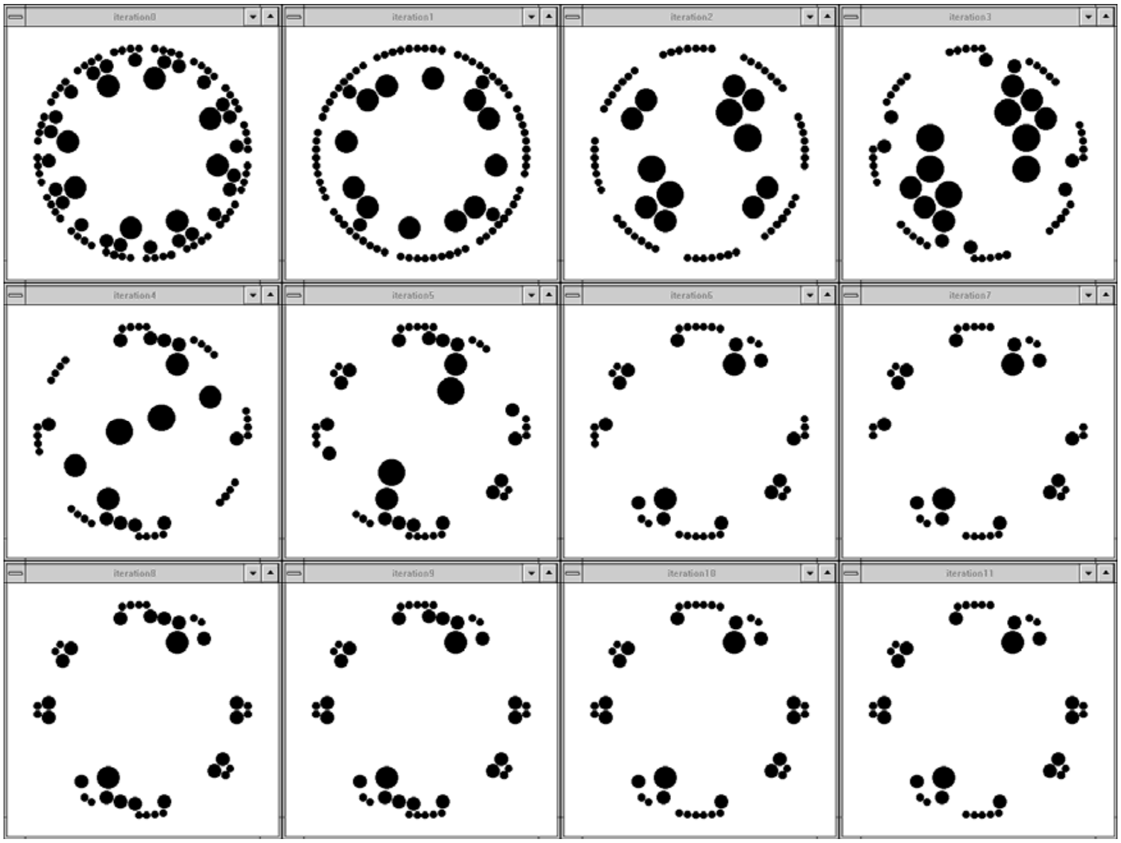

Figure 7. Symmetry breaking, restoration and development of 4-cycles.

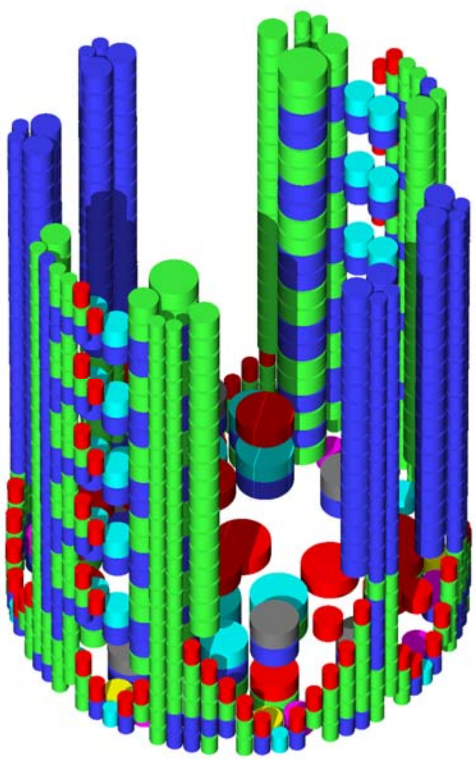

Figure 8. Time evolution showing symmetry breaking, restoration and development of 4-cycles.

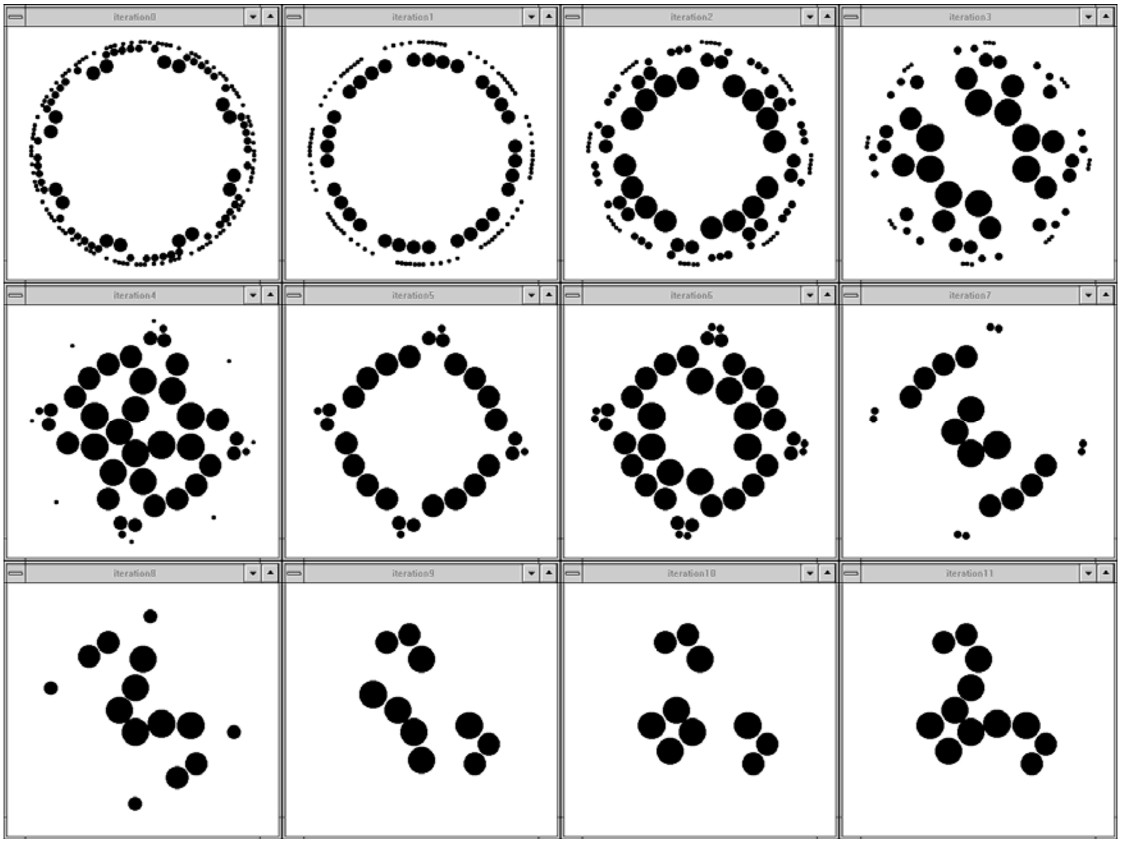

Figure 9. Symmetry breaking and the development of 3-cycles and bilateral symmetry.

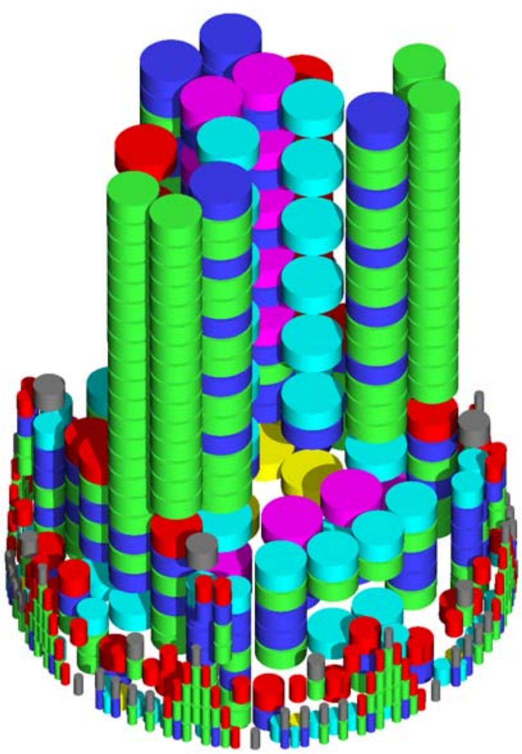

Figure 10. Time evolution showing symmetry breaking and the development of 3-cycles and bilateral symmetry.