Generalized Perfect Parallelograms and Their Matrix Generators(Preprint)

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Abstract

Perfect Parallelograms have edge lengths and diagonal lengths that are all positive integers. These generalize Pythagorean triples which are perfect rectangles. We consider the distribution of perfect parallelograms and show they satisfy a quadratic Diophantine Equation. The solutions to that Diophantine equation can be generated by a finite collection of matrices that generalizes the matrix based tree of Pythagorean Triples.

1   Introduction

A classic open problem in number theory is to determine whether there is a perfect cuboid [3, 5]. That is, is there a rectangular box with edge lengths, face diagonal lengths and body diagonal length all positive integers? In [3] Richard Guy poses the question of whether there is a perfect parallelepiped (a three dimensional parallelepiped with edge lengths, face diagonal lengths and body diagonal lengths all being positive integers). Recently perfect parallelepipeds have been shown to exist
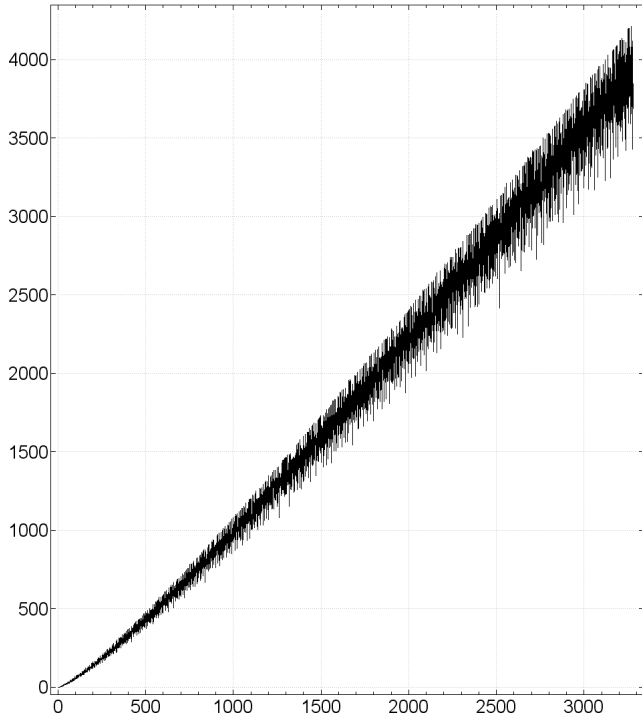
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Figure 1: Number of Perfect Parallelograms with largest edge x_1 versus x_1.

[8]. That paper used assemblies of perfect parallelograms in order to find the perfect parallelepipeds. Those collections of perfect parallelograms were created by focused brute force using an efficient representation and were computed using J[7].

In this paper we explore perfect parallelograms in their own right. Figure 1 shows the number of perfect parallelograms with largest edge x_1 versus x_1. We see that the number is erratic, but there appears to be a definite trend that is perhaps slightly more than linear. Figure 2 shows the number of perfect parallelgrams with largest edge length $x_1 = 2201$ and other edge x_2 versus x_2. For small x_2 no perfect parallelograms exist while for larger x_2 up to 15 different diagonal configurations give perfect parallelograms for the same edge pair. This is quite different from the case of Pythagorean triangles (perfect rectangles) which have at most one solution for a fixed edge pair.

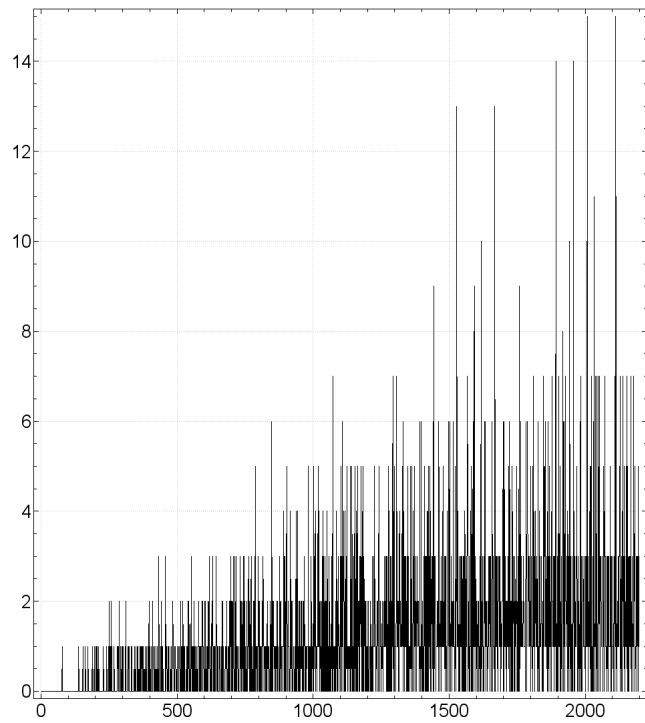Pythagorean triples can be enumerated and described in many ways [6]. One

Figure 2: Number of Perfect Parallelograms with largest edge $x_1 = 2201$ versus x_2.

remarkeable method involves the multiplication of Pythagorean triples by the matrix

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

and sign change variants [1, 2, 4]. Matrix generators for sums of squares in higher dimension [9] have also been found. We define perfect parallelograms as parallelograms that have edge lengths and diagonal lengths that are positive integers. Thus, Pythagorean triples correspond to perfect rectangles. In this note we will see that if we generalize the notion of perfect parallelogram, we have a similar collection of matrix generators. We will also see that perfect parallelograms are the generalized perfect parallelograms that satisfy one additional linear inequality.

2   Preliminaries

The first theorem states that the edge lengths and diagonal lengths of a perfect parallelogram satisfy a Diophantine equation.

Proposition 1. *Let x_1 and x_2 be the lengths of the edges of a parallelogram and let z_1 and z_2 be the lengths of the diagonals. Then $2x_1^2 + 2x_2^2 = z_1^2 + z_2^2$.*

Proof. Let $\vec{u}$ and $\vec{v}$ be edge vectors for the parallelogram so that $\|\vec{u}\| = x_1$, $\|\vec{v}\| = x_2$, $\|\vec{u} - \vec{v}\| = z_1$, and $\|\vec{u} + \vec{v}\| = z_2$. Note that $\|\vec{u} \pm \vec{v}\|^2 = \|\vec{u}\|^2 \pm 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$. Then the result follows from observing that the cross terms cancel: $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$ $\square$

Thus any perfect parallelogram gives a positive integer solution to the Diophantine equation $2x_1^2 + 2x_2^2 = z_1^2 + z_2^2$. For example, $\langle x_1, x_2, z_1, z_2 \rangle = \langle 4, 3, 5, 5 \rangle$ gives a solution corresponding to the $3-4-5$ right triangle. On the other hand, the Diophantine equation has other solutions, for example $\langle x_1, x_2, z_1, z_2 \rangle = \langle 3, -2, 5, -1 \rangle$, that do not correspond to a parallelogram. Thus, we define a *generalized perfect parallelogram* as a list $\langle x_1, x_2, z_1, z_2 \rangle$ of integers, not all zero, such that $2x_1^2 + 2x_2^2 = z_1^2 + z_2^2$. Proposition 1 implies that perfect parallelograms are generalized perfect parallelograms.

If $\langle x_1, x_2, z_1, z_2 \rangle$ is a generalized perfect parallelogram then so is any list where the signs of the coordinates may be independently switched. We can also interchange x_1 with x_2 and/or z_1 with z_2. Thus we say a generalized perfect parallelogram $\langle x_1, x_2, z_1, z_2 \rangle$ is *in standard form* if all the coordinates are nonnegative and $x_1 \geq x_2$

10 9 1 19

9 8 11 13   **11 8 9 17**

9 8 1 17   11 8 3 19

8 7 1 15   10 7 3 17

7 6 7 11

7 6 1 13   11 6 5 17

6 5 1 11   8 5 3 13   **10 5 9 13**

5 4 1 9   **7 4 7 9**   9 4 5 13   11 4 7 15

7 4 3 11

4 3 5 5   8 3 5 11   10 3 7 13

4 3 1 7

3 2 1 5   5 2 3 7   7 2 5 9   9 2 7 11   11 2 9 13

9 2 1 13   11 2 5 15

2 1 1 3   4 1 3 5   6 1 5 7   8 1 7 9   10 1 9 11

8 1 3 11

1 0 1 1   5 0 1 7

Figure 3: Primitive, Odd, Generalized Perfect Parallelograms with Largest Edge $x_1 \leq 11$.

and $z_1 \leq z_2$. Thus, every generalized perfect parallelogram is *equivalent* (up to sign changes and like-coordinate switches) to one in standard form. When a generalized perfect parallelogram $\langle x_1, x_2, z_1, z_2 \rangle$ is in standard form we can call x_1 the larger edge, x_2 the smaller edge (these will be the same for a perfect rhombus), and z_1 is the minor diagonal while z_2 is the major diagonal (these will be the same for a perfect rectangle).

We say a generalized perfect parallelogram is *primitive* if the greatest common divisor of its coordinates is one. Notice that integer solutions to $2x_1^2 + 2x_2^2 = z_1^2 + z_2^2$ must have z_1 and z_2 have the same parity (even/odd). Thus, we say a generalized perfect parallelogram $\langle x_1, x_2, z_1, z_2 \rangle$ is *odd* if z_1 and z_2 are odd and we say it is *even* if z_1 and z_2 are even.

Figure 3 shows the primitive, odd, generalized perfect parallelograms in standard form whose largest edge is 11 or less.

The next theorem gives the condition for a generalized perfect parallelogram to be a perfect parallelogram.

Theorem 2. *Let $\langle x_1, x_2, z_1, z_2 \rangle$ be a generalized perfect parallelogram in standard form, then it is a perfect parallelogram if and only if $x_1 < x_2 + z_1$.*

5

Proof. We see that if $\langle x_1, x_2, z_1, z_2 \rangle$ is a perfect parallelogram in standard form, then the larger edge must be smaller than the sum of the smaller edge with the minor diagonal by the triangle inequality, hence $x_1 < x_2 + z_1$. On the other hand, suppose we have a generalized perfect parallelogram $\langle x_1, x_2, z_1, z_2 \rangle$ in standard form such that $x_1 < x_2 + z_1$. Notice that $x_2 \neq 0$ since if $x_2 = 0$ then the inequality we are presuming becomes $x_1 < z_1$ and the Diophantine equation becomes $2x_1^2 = z_1^2 + z_2^2 \geq 2z_1^2$ since standard form requires $z_2 \geq z_1$. This gives a contradiction; hence, $x_2 \neq 0$ and $x_1 \neq 0$ since $x_1 \geq x_2$. Next we claim that the following inequalities hold.

$$0 \leq \frac{x_1^2 + x_2^2 - z_1^2}{2x_1 x_2} < 1$$

Now $x_1 < x_2 + z_1$ implies $(x_1 - x_2)^2 < z_1^2$ which implies $\frac{x_1^2 + x_2^2 - z_1^2}{2x_1 x_2} < 1$. The Diophantine equation gives $2x_1^2 + 2x_2^2 = z_1^2 + z_2^2 \geq 2z_1^2$ which implies $x_1^2 + x_2^2 - z_1^2 \geq 0$ showing the desired quotient is nonnegative. So the claim is true and thus it is possible to find an angle $0 < \theta \leq \frac{\pi}{2}$ so that we get the following.

$$cos(\theta) = \frac{x_1^2 + x_2^2 - z_1^2}{2x_1 x_2}$$

Now let $\vec{u} = \langle x_1, 0 \rangle$ and $\vec{v} = \langle x_2 \cos\theta, x_2 \sin\theta \rangle$. We see $\|\vec{u}\| = x_1$ and $\|\vec{v}\| = x_2$. Also, $2\vec{u} \cdot \vec{v} = 2x_1 x_2 cos(\theta) = x_1^2 + x_2^2 - z_1^2$ so that

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} = x_1^2 + x_2^2 - (x_1^2 + x_2^2 - z_1^2) = z_1^2$$

which gives $\|\vec{u} - \vec{v}\| = z_1$ and similarly

$$\|\vec{u} + \vec{v}\|^2 = 2x_1^2 + 2x_2^2 - z_1^2 = z_2^2$$

by the Diophantine equation, and so $\|\vec{u} + \vec{v}\| = z_2$. Thus $\vec{u}$ and $\vec{v}$ give the edges of the perfect parallelogram $\langle x_1, x_2, z_1, z_2 \rangle$ as required. $\qquad\square$

3   Matrix Generators

We begin by defining matrices S and E below. Direct computation shows that S^{-1} is the claimed inverse matrix. We will see that muliplication by these matrices preserves generalized perfect parallelograms. The matrices S and E preserve perfect parallelograms too.

$$S = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix} \quad S^{-1} = \begin{pmatrix} 2 & 1 & -1 & -1 \\ 1 & 2 & -1 & -1 \\ -2 & -2 & 2 & 1 \\ -2 & -2 & 1 & 2 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

For example, given $\vec{w} = \langle 2, 1, 1, 3 \rangle$ which is a generalized perfect parallelogram in standard form, then so is $S\vec{w} = \langle 9, 8, 11, 13 \rangle$ while $S^{-1}\vec{w} = \langle 1, 0, -1, 1 \rangle$ which is a generalized perfect parallelogram, but not in standard form. Note that we will freely write vectors either as horizontal lists or column vectors although we will be consistent in any one context. Also, given the perfect parallogram $\vec{w} = \langle 4, 3, 5, 5 \rangle$ then we also get another: $E\vec{w} = \langle 5, 5, 6, 8 \rangle$ and another $S\vec{w} = \langle 21, 20, 29, 29 \rangle$ but $S^{-1}\vec{w} = \langle 1, 0, 1, 1 \rangle$ is only a generalized perfect parallelogram.

Proposition 3. *Let $\vec{w}$ denote a vector of four integers that are not all zero.*
(a) $\vec{w}$ is a generalized perfect parallelogram if and only if $S\vec{w}$ is.
(b) If $\vec{w}$ is a generalized perfect parallelogram in standard form then so is $S\vec{w}$.
(c) If $\vec{w}$ is a perfect parallelogram in standard form then so is $S\vec{w}$.
(d) $\vec{w}$ is a generalized perfect parallelogram if and only if $E\vec{w}$ is.
(e) $\vec{w}$ is a generalized perfect parallelogram in standard form if and only if $E\vec{w}$ is.
(f) If $\vec{w}$ is a perfect parallelogram in standard form then so is $E\vec{w}$.
(g) The map $\vec{W} = E\vec{w}$ gives a bijection between the set of primitive odd generalized perfect parallelograms and the set of primitive even generalized perfect parallelograms.

Proof. Let $\vec{w} = \langle x_1, x_2, z_1, z_2 \rangle$ and $\vec{W} = \langle X_1, X_2, Z_1, Z_2 \rangle$.
(a) Suppose $\vec{W} = S\vec{w}$. Then direct computataion of the Diophantine equation (with all terms on one side) verifies that

$$2X_1^2 + 2X_2^2 - Z_1^2 - Z_2^2 = 2(2x_1 + x_2 + z_1 + z_2)^2 + 2(x_1 + 2x_2 + z_1 + z_2)^2$$
$$- (2x_1 + 2x_2 + 2z_1 + z_2)^2 - (2x_1 + 2x_2 + z_1 + 2z_2)^2$$
$$= 2x_1^2 + 2x_2^2 - z_1^2 - z_2^2$$

and hence $\vec{W}$ is a generalized perfect parallelogram if and only if $\vec{w}$ is one.
(b) Suppose $\vec{W} = S\vec{w}$. Note that $X_1 \geq X_2$ means $2x_1 + x_2 + z_1 + z_2 \geq x_1 + 2x_2 + z_1 + z_2$ which is equivalent to $x_1 \geq x_2$ and $Z_1 \leq Z_2$ means $2x_1 + 2x_2 + 2z_1 + z_2 \leq 2x_1 + 2x_2 + z_1 + 2z_2$ which is equivalent to $-z_2 \leq -z_1$ or $z_1 \leq z_2$. Also, the entries in $\vec{W}$ will be nonnegative if the entries in $\vec{w}$ are, and so we see $\vec{W}$ will be in standard form if $\vec{w}$ is in standard form.

(c) Suppose $\vec{W} = S\vec{w}$. In light of (b) and Theorem 2, it suffices to check that $x_1 < x_2 + z_1$ implies $X_1 < X_2 + Z_1$. Note that $X_2 - X_1 + Z_1 = (x_2 - x_1 + z_1) + (2x_1 + 2x_2 + z_1 + z_2) > 0$ since the left is positive by assumption and the right is nonnegative.

(d) Suppose $\vec{W} = E\vec{w}$. We suppose $\vec{W} = E\vec{w}$. We see the Diophantine equation (with all the terms on one side) is

$$2X_1^2 + 2X_2^2 - Z_1^2 - Z_2^2 = 2z_2^2 + 2z_1^2 - (2x_2)^2 - (2x_1)^2$$
$$= -2(2x_1^2 + 2x_2^2 - z_1^2 - z_2^2)$$

and hence $\vec{W}$ is a generalized perfect parallelogram if and only if $\vec{w}$ is one.

(e) Suppose $\vec{W} = E\vec{w}$. Note that the entries of $\vec{w}$ are nonnegative if and only if the entires in $\vec{W}$ are and that $X_1 \geq X_2$ means $z_2 \geq z_1$ and $Z_2 \geq Z_1$ means $2x_1 \geq 2x_2$ so the result follows.

(f) Suppose $\vec{W} = E\vec{w}$. We are assuming that $\vec{w}$ is a perfect parallelogram in standard form, so we know $x_1 < x_2 + z_1$ and also $z_2 < x_1 + x_2$ by the triangle inequality. Thus $z_2 < x_1 + x_2 < 2x_2 + z_1$ so that $X_1 < Z_1 + X_2$ as required.

(g) First note that if $\vec{w}$ is an odd primitive generalized perfect parallelogram, then $E\vec{w} = \langle z_2, z_1, 2x_2, 2x_1 \rangle$ is even, it is a generalized perfect parallelogram by (d), and it is primitive since z_2 does not have a factor of 2 and any other common prime factor would contradict the primitivity of $\vec{w}$. Given an even primitive generalized perfect parallelogram it would have the form $\langle x_1, x_2, 2s_1, 2s_2 \rangle$ where s_1 and s_2 are nonnegative integers. Both x_1 and x_2 must be odd since the Diophantine equation simplifies to $x_1^2 + x_2^2 = 2s_1^2 + 2s_2^2$ which implies x_1 and x_2 have the same parity (even/odd) but both being even would contradict primitivity. Thus $E\langle s_2, s_1, x_2, x_1 \rangle = \langle x_1, x_2, 2s_1, 2s_2 \rangle$ as required to see the map is onto. It is one-to-one because E is invertible.

$\square$

We next turn to a technical lemma that we will find useful for the proof of our main theorem.

Lemma 4. *Let $\langle x_1, x_2, z_1, z_2 \rangle$ be a primitive generalized perfect parallelogram in standard form, then $x_1 + x_2 \leq z_1 + z_2 \leq 2(x_1 + x_2)$ and equality is only possible for $\langle x_1, x_2, z_1, z_2 \rangle = \langle 1, 0, 1, 1 \rangle$ and $\langle x_1, x_2, z_1, z_2 \rangle = \langle 1, 1, 0, 2 \rangle$.*

Proof. We know the coordinates are non-negative and $2x_1^2 + 2x_2^2 = z_1^2 + z_2^2$. The fact that $(x_1 - x_2)^2 \geq 0$ implies $x_1^2 + x_2^2 \geq 2x_1x_2$ so that $2x_1^2 + 2x_2^2 = x_1^2 + x_2^2 + x_1^2 + x_2^2 \geq x_1^2 + x_2^2 + 2x_1x_2 = (x_1 + x_2)^2$. So we see $(x_1 + x_2)^2 \leq z_1^2 + z_2^2 \leq (z_1 + z_2)^2$. Taking roots gives the desired left-hand inequality: $x_1 + x_2 \leq z_1 + z_2$ where equality can

8

only occur for $z_1 = 0$ and $x_1 = x_2$. The Diophantine equation implies x_1 divides z_2 when $z_1 = 0$ and $x_1 = x_2$ contradicting primitivity unless $x_1 = 1$. So equality on the left occurs only for $\langle x_1, x_2, z_1, z_2 \rangle = \langle 1, 1, 0, 2 \rangle$.

By Proposition 3, $\langle z_2, z_1, 2x_2, 2x_1 \rangle$ is also a generalized perfect parallelogram and it is in standard form. Applying the argument to this generalized perfect parallelogram gives $z_1 + z_2 \leq 2(x_1 + x_2)$ and equality can only hold if $x_2 = 0$ and $z_1 = z_2 = 1$ giving $\langle x_1, x_2, z_1, z_2 \rangle = \langle 1, 0, 1, 1 \rangle$ $\qquad\square$

Our main theorem is that all primitive generalized perfect parallelograms can be produced from the two smallest examples via multiplication by a finite collection of matrices. Like-coordinate interchanges and sign changes can be accomplished with matrix multiplication and we need additionally only multiplication by S. The theorem's proof goes the other way, it shows that given any primitive, generalized, perfect parallelogram in standard form, multiplication by S^{-1} gives a smaller one unless the vector is one of the two special generators. Sign changes and coordinate switches can be used to return the result to standard form and then the process is repeated until one of the two special generators is reached.

Theorem 5. *All primitive generalized perfect parallelograms can be produced from $\vec{w}_1 = \langle 1, 0, 1, 1 \rangle$ or $\vec{w}_2 = \langle 1, 1, 0, 2 \rangle$ by a finite sequence of changes of sign, switches of like coordinates and multiplication by S.*

Proof. Suppose that $\vec{w} = \langle x_1, x_2, z_1, z_2 \rangle$ is a generalized perfect parallelogram and let $\vec{W} = S^{-1}\vec{w}$. We claim that $\|\vec{W}\| < \|\vec{w}\|$ except for the special vectors above where $\|\vec{w}_1\| = \|S^{-1}\vec{w}_1\|$ and $\|\vec{w}_2\| = \|S^{-1}\vec{w}_2\|$. Those two special cases are easy to verify. As noted above, proving this claim suffices to prove the theorem. To prove the claim we consider partitioned arrays and let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} \vec{x} \\ \vec{z} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

$$\vec{W} = \begin{pmatrix} \vec{X} \\ \vec{Z} \end{pmatrix}, \quad \vec{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

The fact that $\begin{pmatrix} \vec{x} \\ \vec{z} \end{pmatrix}$ gives a generalized perfect paralelogram means it satisfies the Diophantine equation and thus $\sqrt{2}\|\vec{x}\| = \|\vec{z}\|$ and $\sqrt{3}\|\vec{x}\| = \|\vec{w}\|$. Therefore to prove the claim about S^{-1} reducing vector sizes, it suffies to show that $\|\vec{X}\| \leq \|\vec{x}\|$ and equality holds only for the two special cases noted above. Now $\vec{W} = S^{-1}\vec{w}$ means

$$\begin{pmatrix} \vec{X} \\ \vec{Z} \end{pmatrix} = \begin{pmatrix} I + N & -N \\ -2N & I + N \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{z} \end{pmatrix}$$

9

so we see that $\vec{X} = (I + N)\vec{x} - N\vec{z} = \vec{x} + N(\vec{x} - \vec{z})$ and therefore

$$\|\vec{X}\|^2 = \|\vec{x}\|^2 + 2\vec{x} \cdot (N(\vec{x} - \vec{z})) + (N(\vec{x} - \vec{z})) \cdot (N(\vec{x} - \vec{z}))$$

and the last term simplifies using $N^T N = 2N$ into

$$(N(\vec{x} - \vec{z}))^T (N(\vec{x} - \vec{z})) = 2(\vec{x} - \vec{z}) \cdot (N(\vec{x} - \vec{z}))$$

Therefore we get
$$\|\vec{X}\|^2 = \|\vec{x}\|^2 + 2(2\vec{x} - \vec{z}) \cdot (N(\vec{x} - \vec{z}))$$
The desired $\|\vec{X}\| \leq \|\vec{x}\|$ will hold once we show the last term is not positive.

$$(2\vec{x} - \vec{z}) \cdot (N(\vec{x} - \vec{z})) = (2(x_1 + x_2) - (z_1 + z_2))(x_1 + x_2 - (z_1 + z_2))$$

In general $(2a - b)(a - b) \leq 0$ if and only if $a \leq b \leq 2a$ so we are done if we can show $x_1 + x_2 \leq z_1 + z_2 \leq 2(x_1 + x_2)$. However, that is true by Lemma 4 with equality holding in the two special cases as required. $\square$

Here is an example. Let $\vec{u}_1 = \langle 53, 44, 51, 83 \rangle$ which is a perfect parallelogram.
$S^{-1}\vec{u}_1 = \langle 16, 7, -9, 23 \rangle \sim \langle 16, 7, 9, 23 \rangle = \vec{u}_2$ where "$\sim$" indicates the vector on the left has been put into standard form using sign changes and like-coordinate interchanges.
$S^{-1}\vec{u}_2 = \langle 7, -2, -5, 9 \rangle \sim \langle 7, 2, 5, 9 \rangle = \vec{u}_3$
$S^{-1}\vec{u}_3 = \langle 2, -3, 1, 5 \rangle \sim \langle 3, 2, 1, 5 \rangle = \vec{u}_4$
$S^{-1}\vec{u}_4 = \langle 2, 1, -3, 1 \rangle \sim \langle 2, 1, 1, 3 \rangle = \vec{u}_5$
$S^{-1}\vec{u}_5 = \langle 1, 0, -1, 1 \rangle \sim \langle 1, 0, 1, 1 \rangle = \vec{w}_1$, which is one of the generators.

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