Chaotic attractors exhibiting quasicrystalline structure

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Abstract
An extension of canonical projection allowing the projection of objects from higher dimensional space onto quasicrystalline structures is developed. In particular, we create symmetric chaotic attractors in 5-dimensionsal space and then project them to the plane such that the resulting image exhibits the structure of a quasicrystalline tiling. These images give a new visual expression of the higher dimensional symmetry of the corresponding attractor.

Keywords: Penrose tilings, quasicrystals, aperiodic tilings, canonical projection

1. Introduction

Penrose tilings are remarkable patterns with no translational repetition but which nonetheless are repetitive in the sense that any finite patch that appears once in the tiling must appear infinitely often. Moreover, Penrose tilings have local 5-fold symmetry which results in a diffraction pattern with 5-fold rotational symmetry — such a diffraction pattern is impossible for patterns which have crystallographic structure. Hence, Penrose tilings are one of the simplest examples of what are now known as quasicrystals. Figure 1 shows an example of a Penrose tiling. Penrose tilings may be created in various ways, but the method of canonical projection is central to our techniques. Figure 2 shows a single 5-dimensional hypercube projected onto a plane where the bold edges correspond to canonical projection. We will describe and generalize canonical projection in Sections 3 and 4, but for now, we note that it is based upon projecting only select lattice points in 5-dimensional space and hence may be viewed as a kind of limited projection from 5-dimensional space.

Chaos is ordinarily associated with a high level of randomness and unpredictability. Symmetry is associated with structure and pattern. Remarkably, these phenomenon may occur simultaneously. Indeed, chaotic attractors with various symmetries have been the subject of considerable recent study because of the intrigue of those seemingly conflicting behaviors and the aesthetic appearance of the results. These studies have included chaotic attractors with cyclic and dihedral symmetry, and also frieze and planar crystallographic symmetry [1-4] along with higher dimensional point groups and general space groups [5-8]. Figure 2 shows a chaotic attractor with cm symmetry that was created in the manner described in [5] and which will be further described in the next section.

In this paper we generalize the canonical projection method; we call the generalization "quasicrytalline projection". It will allow us to project periodic chaotic attractors in 5-dimensional space to the plane in a way that is consistent with the quasicrystalline tiling that results from the corresponding lattice. Our examples offer aesthetically pleasing quasicrytalline patterns that give glimpses into the five dimensional symmetry of the attractor.

2. Symmetric Chaotic Attractors

The symmetry group cm is a planar crystallographic group and hence includes translations by the integer lattice. In the International Tables [9] we see the additional symmetries of cm are generated by $(x+1/2, y+1/2)$ and $(-x, y)$.

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To facilitate the construction of functions with these symmetries, we can represent the symmetries via a single matrix in homogeneous coordinates. The point (x, y) is $(x, y, 1)$ in homogenous coordinates. Thus we see

$$(x, y, 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix} = (x + 1/2, y + 1/2, 1)$$ which represents the first symmetry mentioned above. In a similar way, the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents the second symmetry. We compute the "position group" generated by these symmetries by finding the modified matrix product closure of the generators. The modification required is that we reduce the entries of the last row, except the rightmost "1", (those entries corresponding to translations) modulo 1. This is the form we use in our implementation. Thus, the position group represents the symmetry group modulo the lattice. In the case of the symmetry group cm, there are 4 elements of the position group. For us, the following theorem from [5] tells us how to create functions with any dimensional crystallographic symmetry.

Theorem. *Let P be a position group on a lattice L in* $\Re^n$ *, and let* f *be a periodic function mod L. Then the function* $\mathrm{i}(\vec{x}) = \vec{x} + \sum_{\mathrm{p} \in P} \mathrm{p}^{-1}(\mathrm{f}(\mathrm{p}(\vec{x})))$ mod *L has the translational symmetries of the lattice L and the symmetries of the position group P.*

Thus our strategy for creating chaotic attractors exhibiting quaiscrytalline structure is as follows. We randomly produce finite 5-dimensional Fourier series and select a 5-dimensional position group. We use them to construct the function $\mathrm{i}(\vec{x})$, described in the Theorem, and thus we know the function has the desired symmetries. We numerically test functions created that way to estimate their Ljapunov exponent. When the highest Ljapunov exponent is between 0.01 and 0.3, we render low resolution images of projections of the attractor. Such a Ljapunov exponent is indicative of chaos, but is not so expansive that we expect difficulty in rendering it. We then select the results of promising experiments for further investigation.

3. Penrose Tilings and Canonical Projection

As we remarked earlier, Penrose tilings have local five-fold symmetry but no translational symmetry. Penrose described them in [10] and descriptions of several techniques for constructing them are given in [11-12]. In particular, methods for constructing Penrose tilings include the following:

- they may be constructed using matching rules on two prototiles (which are rhombs);
- they may be created by successive substitution according to rules (whereby the tiles in a tiling are replaced by smaller versions of the themselves);
- they may be constructed via pentagrids (the Penrose tiling is dual to a tiling created by five families of parallel lines); and
- they may be constructed via "canonical" projection from higher dimensional space.

These techniques may be easily generalized to produce quasicrystalline tilings other than the Penrose tilings, but we will describe canonical projection as it would be used to to create a Penrose tiling.

Consider the standard integer lattice of 5-dimensional space tiled by 5-dimensional unit hypercubes. The vector $u = (1,1,1,1,1)$ is fixed by the fifth turn that shifts coordinates, $\vec{r}(\vec{x}) = (\vec{x},1)R$, that is represented by the 6 by 6 homogeneous coordinate matrix $R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. We construct an irrational 2-dimensional subspace *E* perpendicular to *u*; here "irrational" means that the subspace intersects the lattice only at the origin. We let $E^{\perp}$ denote the 3-dimensional subspace of 5-dimensional space that is orthogonal to *E*. For this hypercube lattice, consider the Voronoï cell around the origin. These are the points which are at least as close to the origin as to any other lattice point. Thus, the Voronoï cell around the origin is a 5-dimensional hypercube with vertices $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. We let *C* denote compact set that is the projection of this Voronoï cell onto $E^{\perp}$. Canonical projection is the projection onto *E* of the points of the integer lattice in 5-dimensional space that project onto *C* in $E^{\perp}$. Roughly speaking, the compact set *C* in $E^{\perp}$ corresponds to a thin cylinder in 5-dimensional space that is "near" *E*. Figure 2 shows the Voronoï cell projected onto *E*. The thick edges correspond to those (few) that would be projected under canonical projection and they could form the beginning of a Penrose tiling. We get the tiling in *E* by projecting the points and edges of the 5-dimensional lattice (not just one hypercube) that lie in the cylinder. To be more precise, to get a true Penrose tiling, we need to take an appropriate translational shift of the Voronoï cell to avoid a singular arrangement. The references and our implementation discuss this. Also note that while we have described the projection in terms of 5-dimensional space, canonical projection is more general and, in fact, we have not used particular features of $\vec{r}(\vec{x})$ or the role of 5-dimensional space except to explain what a Voronoï cell would look like in the space.

4. Quasicrystalline Projection

We have seen that the projection method can be used to create quasicrystalline tilings. However, the method is only designed for the projection of select lattice points. We face the difficulty of projecting chaotic attractors which wander around 5-dimensional space. However, our attractors are periodic with the period of the each of the lattice coordinates.

We could identify the vertices in 5-dimensional space that lie in the cylinder and then project each of the points of the attractor added to that (integer) vertex. This induces noticeable visual bias since not every point which is projected under that scheme would have been in the cylinder, nor would every point of the attractor that lies in the cylinder arise that way. That is, the bias comes from missing some points in the cylinder and projecting some outside of the cylinder. The mistakes come in hypercube fractional clumps, which as we see in Figure 2, can be thicker in some regions and thinner in others.

Thus we turn to our generalization of canonical projection, which we call quasicrystalline projection. The goal is simple and compelling: we project only the points of the attractor that lie in the cylinder. Accomplishing this is computationally expensive. We first note that our attractor points have coordinates between 0 and 1. Thus, a point which is in the cylinder will have integer part which is a projected point or some of the coordinates may be one less than the nearest projected lattice point. Thus, we create a list of all the integer lattice points in 5-dimensional space that are in the cylinder and which are close enough to the origin to project onto our finite screen; we add 0 or -1 to each coordinate in all of the 32 possible ways, and then remove duplicates from that extended list. We then add each point in the attractor to each 5-tuple in the extended list and test whether it lies in the cylinder. Testing whether a single point lies in the cylinder requires projecting the point, finding its distance in the direction of the 10 normals to the facets of *C*, and then testing 10 inequalities. If the projection of a point to $E^{\perp}$ does lie in *C*, we project the point to *E*; otherwise, we do not project the point. As we do the projection, we maintain the frequency count of the number of times each pixel is visited. Even though we want to project millions of iterations of the function and there are hundreds of 5-tuples in our extended list, the computation is tractable and we see the results of several experiments in the next section.

5. Examples of Quaiscrystalline Projection

For our illustrations we take *E* to be the subspace of 5-dimensional space generated by: $(\cos(0\pi/5), \cos(2\pi/5), \cos(4\pi/5), \cos(6\pi/5), \cos(8\pi/5))$ and $(\sin(0\pi/5), \sin(2\pi/5), \sin(4\pi/5), \sin(6\pi/5), \sin(8\pi/5))$. The space $E^{\perp}$ is generated by the vectors $(\cos(0\pi/5), \cos(4\pi/5), \cos(8\pi/5), \cos(12\pi/5), \cos(16\pi/5))$ and $(\sin(0\pi/5), \sin(4\pi/5), \sin(8\pi/5), \sin(12\pi/5), \sin(16\pi/5))$ and $(1,1,1,1,1)$. We use a shift of the Voronoï cell by (0, 0, 0.696, 0.879, 1.11803) to avoid a singular arrangement. Those choices, along with the quasicrystalline projection techniques in general, is enough to specify the specific quasicrystalline projection that we are using.

Figure 4 shows our first illustration of a quasicrystalline projection along with the corresponding quasicrystalline tiling. The generators of the position group are given in the appendix and there we also see the position group has 160 elements. Each of the symmetry groups used in our examples include $\vec{r}(\vec{x})$ from Section 3 as a generator. There is no reason that objects must have the symmetry of $\vec{r}(\vec{x})$ in order to be projected with quasicrystalline projection; however, in practice, we found no interesting examples of attractors using quasicrytalline projection on attractors without the symmetry of $\vec{r}(\vec{x})$. This is reasonable since the structure of quasicrstalline tilings is subtle, so having no portions of the object highlight the local symmetry would make it difficult to observe any structure. Notice in Figure 4 the small local 5-fold rotational symmetric regions appear to be scattered and often repeated. The red regions correspond to a regions visited least often while the colors move through the hues toward magenta as the number of visits to the pixel location increases. We use a logarithmic bias toward giving more color change to the higher frequencies; this highlights the detail in the most visited regions. This also results in the prominence of red and orange. The large number of features is consistent with the relatively large position group. Notice that the pattern in any rhomb may differ from the same rhomb in another position unless the rhombs have similar local configurations; in that case, we can observe the consistency.

Figure 5 shows a chaotic attractor exhibiting quasicrystalline structure and also featuring many glides. The position group has 160 elements in this illustration too. Here we have chosen a palette for the image aesthetically; here green corresponds to the regions visited a low number of times. In this image mirrors are apparent; this results in a wandering and changing floral-like pattern.

Figure 6 shows another chaotic attractor exhibiting quasicrystalline structure. The position group has size 10 and is generated by $\vec{r}(\vec{x})$ and a central inversion. This group is as small as is possible if $\vec{r}(\vec{x})$ and any independent symmetry is to be included in the position group. Hence, this attractor is more generic than our other examples. This image has striking swirls and while the inversion adds structure and repetition, there are no apparent local mirrors. Purple corresponds to the regions with a low number of visits.

Figure 7 shows a chaotic attractor exhibiting quasicrystalline structure where the position group involves many mirrors. Indeed, the five directions of parallel highlights are apparent which is quite suggestive of the multigrid construction method for Penrose tilings. It appears like a sky with lights aligned, but still scattered. Here black/brown corresponds to regions visited a small number of times.

A J script that would allow readers to duplicate Figure 6 with a generic palette may be obtained from [13]. A few more of our examples of attractors exhibiting quasicrystalline structure also appear there.

We have seen that quasicrystalline projection allows us to display chaotic attractors in 5-dimensional space in a way so that the underlying quasicrystalline structure and higher symmetry are both visible. These provide examples of aesthetic patterns with little overall structure but with the rich local structure of quasicrystals.

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Appendix: Figure Symmetries and Position Group Closure

Each of the figures of chaotic attractors exhibiting quasicrystalline symmetry has the 5-fold symmetry of the map $\vec{r}(\vec{x})$ that was defined in Section 3. Each of the symmetry groups also has a single additional generator corresponding to the matrices *F4*, *F5*, *F6* and *F7*, given below, where the numbers correspond to the figure numbers. A few J expressions implementing and applying position group closure are given after the matrices. The position group for Figure 4 has 160 elements. The position groups for the remaining figures have size 160, 10, and 80, respectively.

$$F4=\begin{pmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&1&0\\0.5&0&0&0&0&1\end{pmatrix},\ F5=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0.5&0.5&0.5&0&0&1\end{pmatrix},$$

$$F6=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&1\end{pmatrix},\ F7=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

Below we offer J functions for computation of the position group closure required for the construction of symmetry chaotic attractors in the Theorem of Section 2. In particular, `Prods^:_~` applied to a list of generators will produce the position group. We assume the matrix *R* from Section 3 and *F4* above have been defined.

```
   prods=:\:~@~.@(**|)@([,(,/)@:((+/ . *)"2/))
   tr=:(5 6$0),5 1#1 0
   Prods=:[: ~. tr"_ |"2 prods

   #Prods^:_~ R,:F4
160
```

The last result above verifies that the number of elements in the position group associated with Figure 4 is 160.

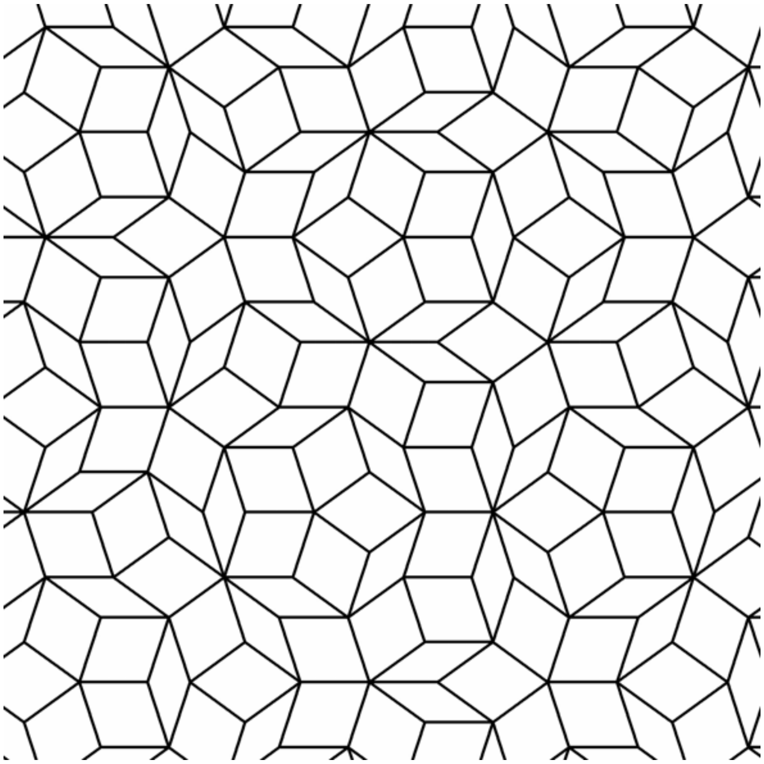

Fig. 1. A Penrose rhomb tiling.

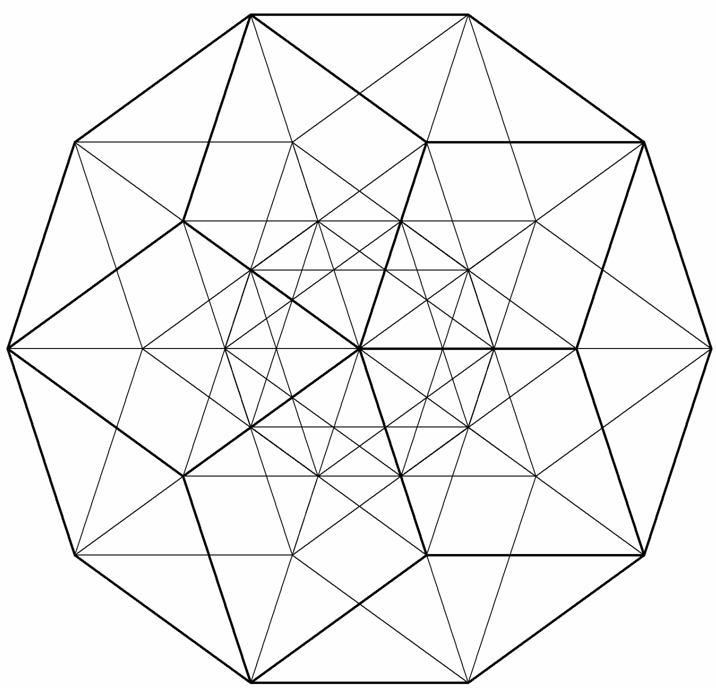

Fig. 2. Projection and canonical projection (bold) of a hypercube.

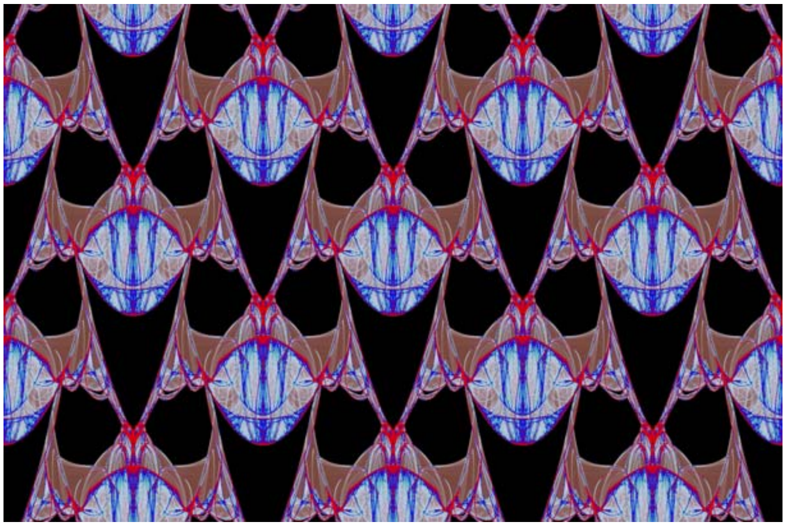

Fig. 3. A chaotic attractor with cm symmetry.

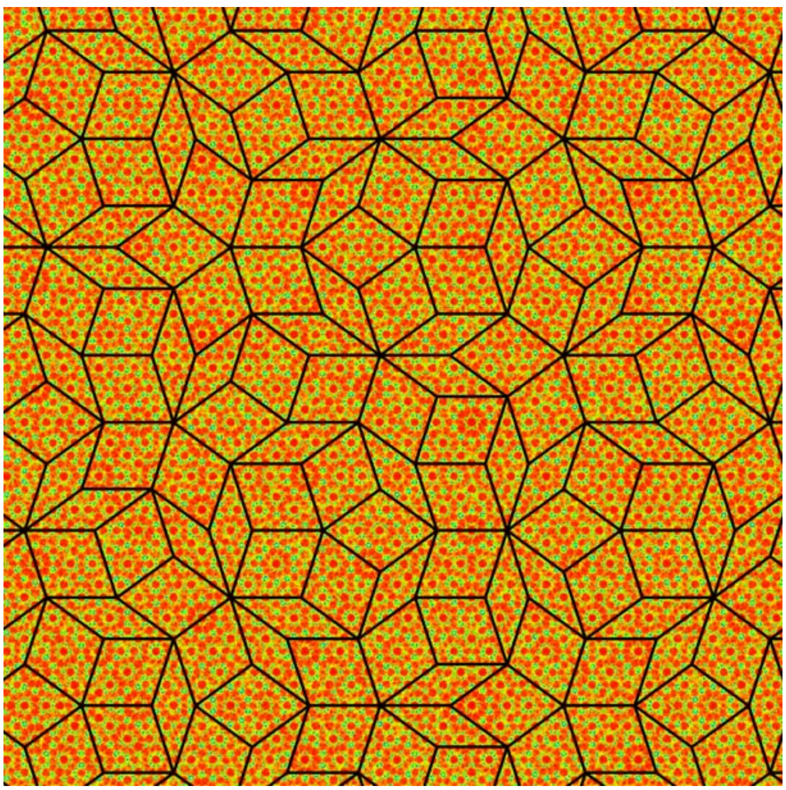

Fig. 4. A chaotic attractor exhibiting quasicrystalline structure and the overlying Penrose tiliing.

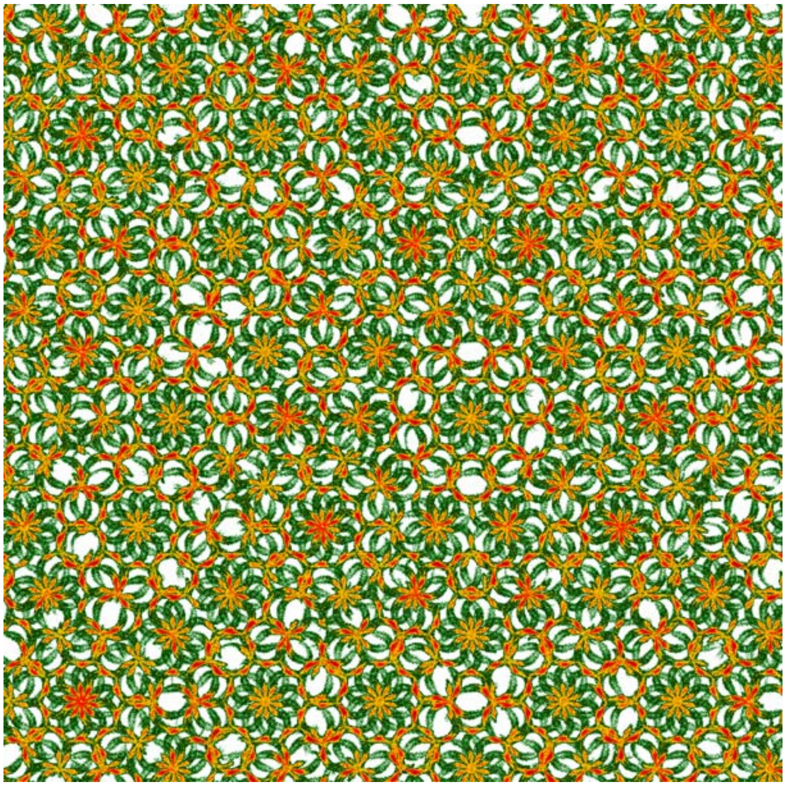

Fig. 5. A chaotic attractor exhibiting quasicrystalline structure and many glides.

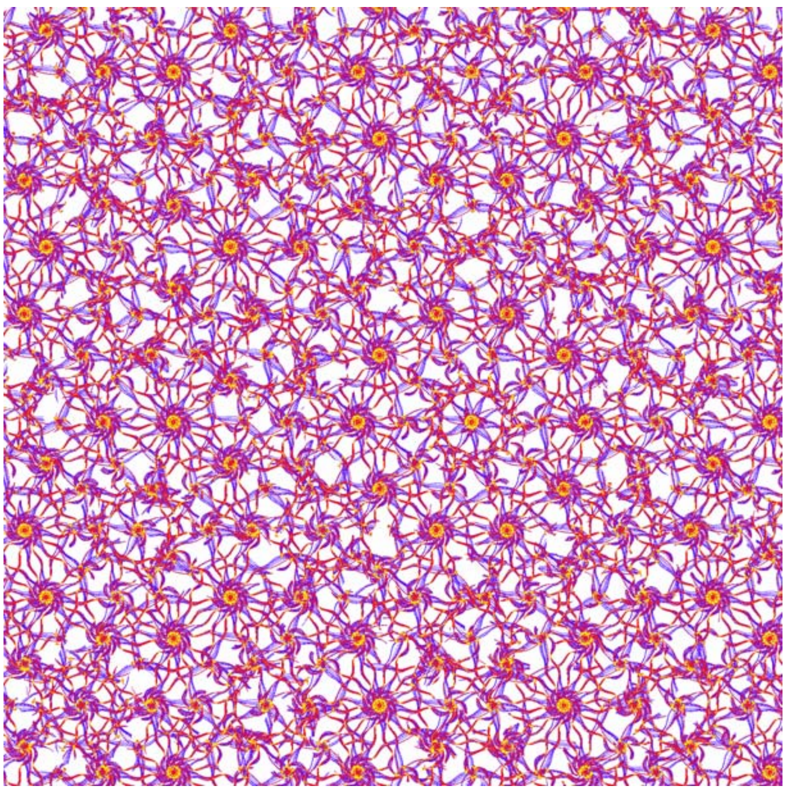

Fig. 6. A chaotic attractor exhibiting quasicrystalline structure and a central inversion.

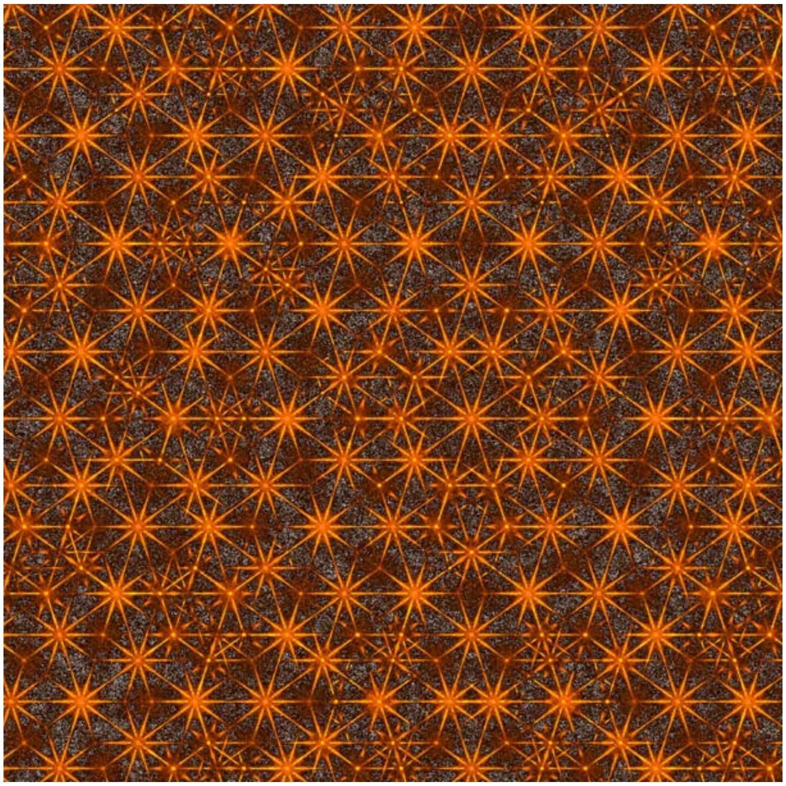

Fig. 7. A chaotic attractor exhibiting quasicrystalline structure and many reflections.