Due: Wednesday, March 7th

1. Show that the following LP is unbounded: Maximize $z = x_1 + 3x_2$ subject to
   
   \[ \begin{align*}
   x_1 - 2x_2 & \leq 4 \\
   -x_1 + x_2 & \leq 3 \\
   x_1, x_2 & \geq 0.
   \end{align*} \]

   Additionally, find feasible solutions so that $z = 5000$ and $z = 10,000$.

2. Use the two-phase Simplex Algorithm to solve the following:
   
   (a) Maximize $z = 3x_1 + x_2$ subject to
       
       \[ \begin{align*}
       x_1 - x_2 & \leq -1 \\
       -x_1 + 5x_2 & \leq -3 \\
       2x_1 + x_2 & \leq 2 \\
       x_1, x_2 & \geq 0.
       \end{align*} \]

   (b) Maximize $z = 4x_1 + 4x_2 + x_3$ subject to
       
       \[ \begin{align*}
       x_1 + x_2 + x_3 & \leq 2 \\
       2x_1 + x_2 & \leq 3 \\
       2x_1 + x_2 + 3x_3 & \geq 3 \\
       x_1, x_2, x_3 & \geq 0.
       \end{align*} \]

   *Note the $\geq$ in the last inequality!

3. Justify the following short answer questions
   
   (a) Is it possible for the auxiliary problem of a two-phase Simplex method to be unbounded? Infeasible? Explain your answer completely.

   (b) If for some LP no table is degenerate, must the simplex algorithm terminate? Explain your answer.

   (c) Assume for some pivot in the solution of an Auxiliary LP you have the option of choosing $x_0$ to exit. Explain why it would be advantageous to do so. Also, in this case, what does this imply about the original LP?

   (d) Consider an LP with 3 variables and 2 constraints. Assuming the simplex algorithm terminates and finds an optimal value, what is the maximum number of iterations needed for the Simplex algorithm to find the optimal value? (Hint: What did we prove about two tables with the same basis?)
4. Assume you know that

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n + D = 0 \]

for any choice of \( x_1, \ldots, x_n \) where \( a_1, \ldots, a_n \) are constants. What can you say about the coefficients \( a_i \) and the constant \( D \)? [Suggestion: Consider the argument in class used to show that if two tables have the same basis, then they are the same table.]

5. In class we said that degenerate solutions arise when we have a choice between two or more exiting variables. This problem will show you that this is the ONLY way such degeneracy occurs.

To illustrate consider the following general non-degenerate table so that \( a_0, b_0 > 0 \):

\[
\begin{align*}
x_4 &= a_0 - a_1x_1 + a_2x_2 \\
x_5 &= b_0 - b_1x_1 + b_2x_2 \\
z &= c_0 + c_1x_1 + c_2x_2
\end{align*}
\]

Now assume that \( a_1, b_1 > 0 \) and that the next table is obtained by having \( x_1 \) enter and \( x_4 \) exit. Show that if this next table is degenerate then we could have also chosen \( x_5 \) instead of \( x_4 \) as the exit variable.

6. Assume you are given an LP with the following two constraints:

\[ f_1(x_1, \ldots, x_n) \leq b_1 \]
\[ f_2(x_1, \ldots, x_n) \leq b_2 \]

Solving the auxiliary LP you discover that its optimal value is \( w = 1 \).

(a) What does this mean about the given LP?

(b) Must any LP with the following constraints be feasible?

\[ f_1(x_1, \ldots, x_n) \leq b_1 + 1 \]
\[ f_2(x_1, \ldots, x_n) \leq b_2 + 1. \]

[Suggestion: Consider the optimal feasible solution to the auxiliary LP for the LP in (2).]