PROBLEMS

PROPOSALS

To be considered for publication, solutions should be received by July 1, 2012.

1886. *Proposed by Jodi Gubernat and Tom Beatty, Florida Gulf Coast University, Fort Myers, FL.*

For which positive integers n is the function value

$$f(n) = \sum_{k=\lfloor n/2 \rfloor}^{n} \left(1 - \frac{2k}{n}\right)^2 \binom{n}{k}$$

an integer?

1887. *Proposed by Elias Lampakis, Kiparissia, Greece.*

Given a circle $\mathcal{C}$ with center O and radius r, and a point H such that $0 < OH < r$,

(a) Show that there are an infinite number of triangles inscribed in $\mathcal{C}$ with orthocenter H.

(b) Determine the set of points belonging to the interior of all triangles inscribed in $\mathcal{C}$ with orthocenter H.

Math. Mag. **85** (2012) 61–68. doi:10.4169/math.mag.85.1.61.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1888. *Proposed by Alex Aguado, Duke University, Durham, NC.*

Let $A \subseteq X$ be a subset of a topological space, and let $N(A)$ denote the number of sets obtained from A by alternately taking closures and complements (in any order). It is well known that $N(A)$ is at most 14. However, exactly for which $r \leq 14$ is it possible to find A and X such that $N(A) = r$?

1889. *Proposed by Gary Gordon and Peter McGrath, Lafayette College, Easton, PA.*

For every positive integer k, consider the series

$$S_k = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}\right) - \left(\frac{1}{3k+1} + \frac{1}{3k+2} + \cdots + \frac{1}{4k}\right) + \cdots .$$

Thus $S_1 = \log 2$ and $S_2 = (\pi + 2\log 2)/4$.

(a) Prove that S_k converges for all k.

(b) Prove that

$$S_k = \int_0^1 \frac{x^k - 1}{(x^k + 1)(x - 1)}\, dx.$$

(c) Prove that the sequence $\{S_k\}$ is monotonically increasing and divergent.

1890. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, Bronx, NY.*

Let m and n be positive integers. Prove that there exist an integer k and a prime p such that $m \equiv k^2 + p \pmod{n}$.

Quickies

Answers to the Quickies are on page 68.

Q1017. *Proposed by Allan Berele and Jeffery Bergen, Department of Mathematics, DePaul University, Chicago, IL.*

Find a monic polynomial $f(x)$ with integer coefficients such that $f(x) = 0$ has no integer solutions but $f(x) \equiv 0 \pmod{p}$ has a solution for every prime p.

Q1018. *Proposed by Finbarr Holland, School of Mathematical Sciences, University College Cork, Cork, Ireland.*

Suppose $0 < \alpha \leq 1$. Prove that

$$e^x \leq \frac{1 + (1-\alpha)x}{1 - \alpha x}$$

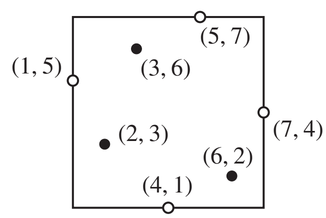
for all $x \in [0, 1)$ if and only if $\alpha \geq 1/2$.

Solutions

Permutations and their bounding squares **February 2011**

1861. *Proposed by Emeric Deutsch, Polytechnic Institute of New York University, Brooklyn, NY.*

Let $n \geq 2$ be an integer. A permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ can be represented in the plane by the set of n points $P_\sigma = \{(k, \sigma(k)) : 1 \leq k \leq n\}$. The smallest square bounding P_σ, with sides parallel to the coordinate axis, has at least 2 and at most 4 points of P_σ on its boundary. The figure below shows a permutation σ with 4 points on its bounding square. For every $m \in \{2, 3, 4\}$, determine the number of permutations σ of $\{1, 2, \ldots, n\}$ having m points of P_σ on the boundary of their bounding square.



Solution by Daniele Degiorgi, ETH Zurich, Zurich, Switzerland.

Let N_m be the number of permutations of $\{1, 2, 3, \ldots, n\}$ having m points of P on the boundary of their bounding square. All the points on the bounding square have 1 or n as x or y coordinate. There are only two points on the boundary if and only if P_σ contains both $(1, 1)$ and (n, n) or both $(1, n)$ and $(n, 1)$. Fixing the two points, none of the $(n-2)!$ permutations of $\{2, 3, \ldots, n-1\}$ will generate points on the boundary. Thus $N_2 = 2(n-2)!$.

There are four points on the boundary if P_σ does not contain any of the four points cited above $(1, 1)$, $(1, n)$, $(n, 1)$, and (n, n). To ensure this, we can select any of the $n-2$ values in $\{2, 3, \ldots, n-1\}$ for $\sigma(1)$, any of the $n-3$ values in $\{2, 3, \ldots, n-1\}$ excluding $\sigma(1)$ for $\sigma(n)$, and any of the $(n-2)!$ possibilities for the other values of of σ. Thus $N_4 = (n-2)(n-3)(n-2)!$.

Finally, $N_3 = n! - N_2 - N_4 = (4n-8)(n-2)!$.

Also solved by Michael Andreoli, Michel Bataille (France), Berry College Dead Poets Society, Jany C. Binz (Switzerland), Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Mark Bowron, Robert Calcaterra, Cal State LA Math Problem Solving Group, John Christopher, CMC 328, Con Amore Problem Group (Denmark), Calvin A. Curtindolph, Patrick Devlin, Daniel Dominik, Gregory Dresden, Dave Feil, Dmitry Fleischman, Natacha Fontes-Merz, David Getling (Germany), Arup Guha, Joshua Ide, Lucyna Kabza, Omran Kouba (Syria), Victor Y. Kutsenok, Elias Lampakis (Greece), László Lipták, Peter McPolin (Northern Ireland), David Nacin, Rituraj Nandan, Northwestern University Math Problem Solving Group, Rob Pratt, Joel Schlosberg, Thomas Q. Sibley, Skidmore College Problem Group, John H. Smith, Philip Straffin, Marian Tetiva (Romania), R. S. Tiberio, Texas State University Problem Solving Group, Dennis Walsh, Michael Woltermann, and the proposer.

Majorization: sum implies product **February 2011**

1862. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let n be a positive integer. Suppose that the nonnegative real numbers $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ satisfy that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$ for all $1 \leq k \leq n$. Prove that $\prod_{i=1}^k a_i \geq \prod_{i=1}^k b_i$ for all $1 \leq k \leq n$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The desired inequalities are trivially true if $a_1 = 0$, because $a_1 = 0$ implies $b_1 = 0$, so let us suppose that $a_1 > 0$.

For every nonnegative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, we have that

$$\sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{k} b_i \right) \le \sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{k} a_i \right).$$

This is equivalent to

$$\sum_{i=1}^{n} \Lambda_i b_i \le \sum_{i=1}^{n} \Lambda_i a_i,$$

where $\Lambda_i = \lambda_i + \lambda_{i+1} + \cdots + \lambda_n$.

Now consider $k \in \{1, 2, \ldots, n\}$ and suppose that $\Lambda_k > 0$. By the previous inequality and the Arithmetic Mean–Geometric Mean Inequality, it follows that

$$\left(\prod_{i=1}^{k} b_i \right)^{1/k} = \frac{1}{\sqrt[k]{\Lambda_1 \cdots \Lambda_k}} \left(\prod_{i=1}^{k} \Lambda_i b_i \right)^{1/k} \le \frac{1}{\sqrt[k]{\Lambda_1 \cdots \Lambda_k}} \cdot \frac{1}{k} \sum_{i=1}^{k} \Lambda_i b_i$$
$$\le \frac{1}{\sqrt[k]{\Lambda_1 \cdots \Lambda_k}} \cdot \frac{1}{k} \sum_{i=1}^{k} \Lambda_i a_i.$$

Choose the λ_i as follows:

$$\lambda_i = \begin{cases} 0 & \text{if } k < i \le n, \\ 1/a_k & \text{if } i = k, \\ 1/a_i - 1/a_{i+1} & \text{if } 1 \le i < k. \end{cases}$$

Note that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nonnegative because $a_1 \le a_2 \le \cdots \le a_k$ and moreover $\Lambda_i = 1/a_i$ for $1 \le i \le k$, and $\Lambda_i = 0$ for $k < i \le n$. Thus

$$\left(\prod_{i=1}^{k} b_i \right)^{1/k} \le \frac{1}{\sqrt[k]{\frac{1}{a_1} \cdots \frac{1}{a_k}}} \cdot \frac{1}{k} \sum_{i=1}^{k} 1 = \left(\prod_{i=1}^{k} a_i \right)^{1/k},$$

which is the desired inequality for k.

Editor's Note. Several solvers pointed out that the inequality is a special case of Karamata's Inequality and involves the idea of majorization. A non-increasing n-tuple $a = (a_1, a_2, \ldots, a_n)$ majorizes another non-increasing n-tuple $b = (b_1, b_2, \ldots, b_n)$ if $\sum_{i=1}^{k} a_i \ge \sum_{i=1}^{k} b_i$ for $k = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. One form of Karamata's Inequality states that if $\phi(t)$ is a continuous, convex function, and a majorizes b, then

$$\sum_{i=1}^{n} \phi(a_i) \ge \sum_{i=1}^{n} \phi(b_i).$$

By choosing $\phi(t) = \log(t)$ and rearranging the sequences, the solution is a special case of Karamata's Inequality.

Also solved by Elton Bojaxhiu (Germany) and Enkel Hysnelaj(Australia); Robert Calcaterra; Marian Dinča (Romania); Robert L. Doucette; John N. Fitch; Vikram Govindan; Lixing Han; Peter W. Lindstrom; László Lipták; Larry A. Lucas, Faryal Bokhari, and Touissant Towa; Peter McPolin (Northern Ireland); Paolo Perfetti (Italy); Joel Schlosberg; Marian Tetiva (Romania); and the proposer.

Moments, $||f'||_\infty$, and Newton–Cotes formulae **February 2011**

1863. *Proposed by Duong Viet Thong, Department of Economics and Mathematics, National Economics University, Hanoi, Vietnam.*

Let f be a continuously differentiable function on $[a, b]$ such that $\int_a^b f(x)\,dx = 0$. Prove that

$$\left|\int_a^b xf(x)\,dx\right| \le \frac{(b-a)^3}{12}\max\{|f'(x)| : x \in [a,b]\}.$$

Solution by Sanghun Song, Seoul Science High School, Jongro-ku, Seoul, Korea.

Let $M = \max\{|f'(x)| : x \in [a,b]\}$, c the midpoint of the interval $[a, b]$, and ℓ its semi-length, i.e., $c = \frac{1}{2}(a+b)$ and $\ell = \frac{1}{2}(b-a)$. Since $\int_a^b f(x)dx = 0$, we have that

$$\int_a^b xf(x)\,dx = \int_a^b (x-c)f(x)\,dx = \int_a^c (x-c)f(x)\,dx + \int_c^b (x-c)f(x)\,dx.$$

Using the change of variable $t = |x - c|$ gives

$$\begin{aligned}\int_a^b xf(x)\,dx &= -\int_0^\ell tf(c-t)\,dt + \int_0^\ell tf(c+t)\,dt \\ &= \int_0^\ell t(f(c+t) - f(c-t))\,dt.\end{aligned}$$

Now, the Mean Value Theorem implies that for every $t \in [0, \ell]$ there is $\xi_t \in (c-t, c+t)$ such that $|f(c+t) - f(c-t)| = |(c+t) - (c-t)||f'(\xi_t)| \le 2tM$. Hence, putting these two facts together we obtain

$$\left|\int_a^b xf(x)\,dx\right| \le \int_0^\ell t|f(c+t) - f(c-t)|\,dt \le 2M\int_0^\ell t^2\,dt = \frac{2\ell^3}{3}M,$$

which is exactly the required inequality.

Editor's Note. Many solutions were based, more or less, on variations of the above argument. Some solvers used a different idea, starting with the twice differentiable function $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$, they observed that $\int_a^b xf(x)\,dx = -\int_a^b F(x)\,dx$, and then used the *l*ocal error for the trapezoid rule also known as one of the Newton–Cotes formulae,

$$\int_a^b F(x)\,dx = (F(a) + F(b))\ell - \frac{2\ell^3}{3}F''(c) \text{ for some } c \in (a,b).$$

Another idea used by R. Calcaterra and D. Dominik was to show that the functional $f \to \int_a^b xf(x)\,dx$ attains its extreme values on the convex set $\{f \in C^1[a,b] : \int_a^b f(x)\,dx = 0$, and $||f'||_\infty \le 1\}$ only for the linear cases, i.e., $f(x) = \pm(x-c)$ with $c \in [a, b]$. Finally it was brought to our attention by P. Perfetti that one can use

Problem E2155 in *American Mathematical Monthly* (December 1969, pp. 1142–1143) to prove the following generalization:

$$\left|\int_a^b x^n f(x)\,dx\right| \le \frac{(n!)^2(b-a)^{2n+1}}{(2n!)(2n+1)!}M,$$

for all f such that $\int_a^b x^k f(x)\,dx = 0$ for $k = 0, 1, 2, \ldots, n-1$.

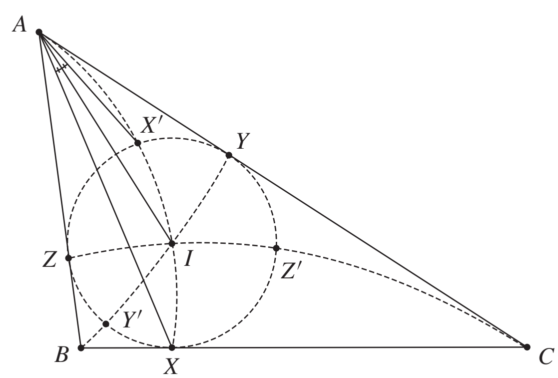
Other connections with the proposed problem can be found in Ch. XV of the book *Inequalities involving functions and their integrals and derivatives* written by D. S. Mitrinović, J. E. Pečaric, and A. M. Fink and printed by Kluwer Academic Publishers, 1991.

Also solved by Armstrong Problem Solvers; Cody M. Allen and William R. Green; Michel Bataille (France); Dionne Bailey, Elsie Campbell, and Charles Diminnie; Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia); Michael W. Botsko; Robert Calcaterra; Hongwei Chen; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal (Spain); Daniel Dominik; Robert L. Doucette; Josh Eyler; Omran Kouba; Charles Lindsay; Peter W. Lindstrom; Rick Mabry; Raymond Mortini (France); Scott Pauley, Andrew Welter, and Natalya Weir; Paolo Perfetti (Italy); Ángel Plaza (Spain); Rob Pratt; Henry Ricardo; Joel Schlosberg; Allen Stenger; Richard Stephens; Xiao Tingben (China); Texas State University Problem Solving Group; Haohao Wang, Jerzy Wojdyło, and Yanping Xia; Luyuan Yu (China); and the proposer. There were three incorrect submissions.

Isogonal conjugate concurrent cevians February 2011

1864. *Proposed by Cosmin Pohoata, Princeton University, Princeton, NJ.*

Let ABC be a scalene triangle, I its incenter, and X, Y, and Z the tangency points of its incircle $\mathcal{C}$ with the sides BC, CA, and AB, respectively. Denote by $X' \neq X$, $Y' \neq Y$, and $Z' \neq Z$ the intersections of $\mathcal{C}$ with the circumcircles of triangles AIX, BIY, and CIZ, respectively. Prove that the lines AX', BY', and CZ' are concurrent.



Solution by Elton Bojaxhiu, Kriftel, Germany, and Enkel Hysnelaj, University of Technology, Sydney, Australia.

Because $IX = IX'$ and A, X', I, and X are cyclic, it follows that $\angle XAI = \angle IAX'$. And since AI is the bisector of $\angle BAC$, we conclude that the lines AX and AX' are symmetrical with respect to this bisector line. Similarly the lines BY' and BY are symmetrical with respect to the line BI, and the lines CZ' and CZ are symmetrical with respect to the line CI.

It is a well-known direct application of Ceva's Theorem that AX, BY, and CZ are concurrent. Since the lines AX', BY', and CZ' are the symmetrical of AX, BY, and CZ with respect to the bisectors AI, BI, and CI, respectively, this implies that the lines AX', BY', and CZ' are concurrent. This fact also follows directly from Ceva's Theorem.

Editor's Note. The intersection of the lines AX, BY, and CZ is called the Gergonne point of $\triangle ABC$. Joel Schlosberg points out that the intersection of the lines AX', BY', and CZ' is the isogonal conjugate of the Gergonne point, and is referred to as the Insimilicenter of the Circumcircle and the Incircle (point X(55)) in Clark Kimberling's Encyclopedia of Triangle Centers available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html. Omran Kouba notes that the required point of concurrency lies on the Euler line of $\triangle XYZ$. Finally, the details of the last part of the proof can be obtained from the trigonometric version of Ceva's Theorem, i.e., AX', BY', and CZ' are concurrent if and only if

$$\frac{\sin\angle BAX' \cdot \sin\angle CBY' \cdot \sin\angle ACZ'}{\sin\angle X'AC \cdot \sin\angle Y'BA \cdot \sin\angle Z'CB} = 1.$$

Also solved by Michel Bataille (France), John G. Heuver (Canada), Omran Kouba (Syria), Joel Schlosberg, Ercole Suppa (Italy), and the proposer.

Reducing representations of a ring **February 2011**

1865. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, Bronx, NY.*

In the solution to Problem 1790 (this *Magazine* **82** (2009) 67–68), it was proved that if R is a ring such that for each element $x \in R$,

$$x + x^2 + x^3 + x^4 = x^{11} + x^{12} + x^{13} + x^{28},$$

then for each element $x \in R$, $x = x^{127}$. Under the same hypothesis, prove that for each element $x \in R$, $6x = 0$ and $x = x^7$.

Solution by Robert Calcaterra, University of Wisconsin–Platteville, Platteville, WI.

Let $f(x) = x^{28} + x^{13} + x^{12} + x^{11} - x^4 - x^3 - x^2 - x$. Fix $c \in R \setminus \{0\}$. Since the zero element obviously satisfies the required conclusions, the proof will be complete if we show that $6c = 0$ and $c^7 = c$.

Let T be the set of all polynomial expressions in c with integer coefficients and constant term 0, i.e., $T = c\mathbb{Z}[c]$. Observe that T is a commutative ring. Moreover, if $e = c^{27} + c^{12} + c^{11} + c^{10} - c^3 - c^2 - c$, then $ce = c$ and so the ring T has unity e. Let $g(x) \in x\mathbb{Z}[x]$. If we divide $g(x)$ by $f(x)$ using the division algorithm, then the reminder $r(x)$ will have the property that $g(c) = r(c)$ because $f(c) = 0$. Consequently, every element of T is an integer combination of $\{c, c^2, c^3, \ldots, c^{27}\}$.

Let n be the order of e in the additive group of T. Assume n is infinite. Then $S := \{ke : k \in \mathbb{Z}\}$ is a subring of T that is isomorphic to the ring of integers. Since S can be embedded in a field, the number of roots of f in S cannot exceed 28. This contradiction forces us to conclude that n is finite. This conclusion further implies that the order of every element in the group $(T, +)$ is a divisor of n. Therefore, the number of distinct elements of T is at most n^{27}. In particular T is a finite ring.

If $b \in R$ and $b^2 = 0$, then $f(b) = 0$ implies that

$$b = b^2(b^{26} + b^{11} + b^{10} + b^9 - b^2 - b) - b^2 = 0.$$

Hence, zero is the only element of R whose square is zero. Therefore, by the Wedderburn–Artin Theorem, T is isomorphic to a finite direct sum of full matrix rings over division rings. Moreover, since T is commutative, these matrices must be 1 by 1 and the components must come from a field. In other words, T must be isomorphic to a direct sum of finite fields.

Let $\mathbb{F}$ be a subfield of T that has q elements (q is a power of a prime). Lagrange's Theorem implies that every element of $\mathbb{F}$ is a root of $x^q - x$. All the roots of this polynomial are simple and so $x^q - x$ must be a divisor of $f(x)$ in the ring $\mathbb{F}[x]$. Therefore, we may use routine computation to show that q is 2, 3, or 4. Hence the characteristic of $\mathbb{F}$ is either 2 or 3 and thus $6x = 0$ for all $x \in \mathbb{F}$. In addition, the order of an element of $(\mathbb{F} \setminus \{0\}, \cdot)$ must be either 1, 2, or 3 and so $x^7 = x$ for all $x \in \mathbb{F}$. Since T is a direct sum of such fields, it follows that $6c = 0$ and $c^7 = c$. This completes the proof.

Note. If $GF(q)$ denotes the field with q elements, then the ring $GF(3) \oplus GF(4)$ meets the hypothesis of the problem and the equation $x^k = x$ is not satisfied by at least one element of this ring when $k < 7$ is a positive integer. Hence 7 is the least positive integer for which the conclusion of this problem is true.

Also solved by John Riegsecker and the proposer.

Answers

Solutions to the Quickies from page 62.

A1017. In $\mathbb{Z}_p^*$, the multiplicative group of non-zero elements of $\mathbb{Z}_p$, the squares form a subgroup of index 2 and so the product of any two non-squares is a square. Hence, at least one of -1, 2 or -2 will be a square. Thus one polynomial that satisfies the requirements is

$$f(x) = (x^2 + 1)(x^2 - 2)(x^2 + 2).$$

A1018. If the inequality is true, then so is the following statement:

$$\frac{e^x - x - 1}{x(e^x - 1)} \le \alpha$$

for all $x \in (0, 1)$. But applying L'Hôpital's rule twice gives

$$\alpha \ge \lim_{x \to 0^+} \frac{e^x - x - 1}{x(e^x - 1)} = \lim_{x \to 0^+} \frac{e^x - 1}{xe^x + e^x - 1} = \lim_{x \to 0^+} \frac{e^x}{e^x(x + 2)} = \frac{1}{2}.$$

Thus, the condition on α is necessary. Conversely, if $\alpha \ge 1/2$, then $1/n! \le 1/2^{n-1} \le \alpha^{n-1}$ for every positive integer n, and so, if $0 \le x < 1$, then

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \le 1 + \sum_{n=1}^{\infty} \alpha^{n-1} x^n = \frac{1 + (1 - \alpha)x}{1 - \alpha x},$$

as required, with equality if and only if $x = 0$.