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On Brylawski's Generalized Duality

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Abstract We introduce a notion of duality—due to Brylawski—that generalizes matroid duality to arbitrary rank functions. This allows us to define a generalization of the matroid Tutte polynomial. This polynomial satisfies a deletion-contraction recursion, where deletion and contraction are defined in this more general setting. We explore this notion of duality for greedoids, antimatroids and demi-matroids, proving that matroids correspond precisely to objects that are simultaneously greedoids and "dual" greedoids.

1 Introduction

Duality plays a central role in graph theory and matroid theory. While only planar graphs have graphic duals, all matroids have duals. Since graphs are matroids, and since geometric duality for planar graphs coincides with matroid duality when the graph is planar, we can view matroid duality as a way to define duals for non-planar graphs.

A matroid can be described by its rank function, and this rank function can then be used to define the three operations of deletion, contraction and duality. T. Brylawski (private communication) realized it is possible to extend all three of these operations to arbitrary "rank" functions. If S is a finite set and $r : 2^S \to \mathbb{Z}$ is any function, then one can define a dual structure via a dual rank function r^*:

$$r^*(A) = |A| + r(S - A) - r(S).$$

If we write $G = (S, r)$ for the finite ground set S with rank function r, then deletion of an element in G can be defined as a restriction of the rank function r:

$$r_{G-p} = r|_{G-p}.$$

Then contraction in G can be defined using deletion and duality:

$$G/p = (G^* - p)^*.$$

This extends the idea of duality to non-matroidal structures. These definitions (along with some background on matroids) and basic results are given in Sect. 2. While this generalized duality is difficult to interpret combinatorially (in particular, we can have $r^*(A) < 0$ for a subset A), we can prove several generalizations of well-known formulas involving deletion, contraction and duality.

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Others have worked on various characterizations of matroid duality. Kung [10] shows that matroid duality is the *only* involution on the class of matroids that interchanges deletion and contraction: $(G-p)^* = G^*/p$ and $(G/p)^* = G^* - p$. Kung's approach is generalized by Whittle [14], who extends duality to *k-polymatroids*, where the dual rank function satisfies $f^*(A) = k|A| + f(S-A) - f(S)$. This also interchanges deletion and contraction, and has the involution property $f^{**} = f$. Bland and Dietrich [2] also investigate duality via generalized involutions, especially for the class of oriented matroids.

Our primary motivation in this work is the close relationship between duality and the Tutte polynomial, the subject of Sect. 3. When M is a matroid, the Tutte polynomial $T(M; x, y)$ is usually defined in one of two equivalent ways: via a subset expansion, or inductively, through a deletion-contraction recursion. The subset expansion uses the rank function:

$$T(M; x; y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)},$$

and this will allow us to define a "Tutte polynomial" in our more general setting.

The matroid version of the deletion–contraction recursion is the following:

$$T(M; x, y) = T(M-p; x, y) + T(M/p; x, y).$$

This formula is generalized in Theorem 3.1(1) in Sect. 3:

$$f(G; t, z) = t^{r(G)-r(G-p)} f(G-p; t, z) + z^{1-r(p)} f(G/p; t, z),$$

where the change of variables $t = x-1$ and $z = y-1$ connects the two formulations. Theorem 3.1(1) also generalizes the deletion-contraction formula for the Tutte polynomial of a greedoid (Proposition 2.5 of [6]). Theorem 3.1(2) shows this general Tutte polynomial is well-behaved with respect to generalized duality (assuming $r(\emptyset) = 0$):

$$f(G^*; t, z) = f(G; z, t).$$

Greedoids are generalizations of matroids, and there are many interesting combinatorial structures that have meaningful interpretations as greedoids, but not matroids. (For instance, trees form greedoids, but the matroid associated with a tree is trivial.) In Sect. 4, we examine our generalized duality for greedoids. The main results are a characterization of the rank axioms dual greedoids satisfy (Theorem 4.3) and a result that shows $\mathcal{G} \cap \mathcal{G}^* = \mathcal{M}$, where $\mathcal{G}$ is the class of all greedoids, $\mathcal{G}^*$ is the class of all greedoid *duals*, and $\mathcal{M}$ is the class of all matroids (Theorem 4.4).

In Sect. 5, we conclude by considering applications to *antimatroids* (a well-studied class of greedoids) and *demi-matroids*, another generalization of matroids [3]. For antimatroids, we interpret the dual rank combinatorially in terms of *convex closure* (Theorem 5.2). For demi-matroids, we examine the connection between our generalized duality and these objects, characterizing precisely the properties the rank function r must satisfy to produce a demi-matroid (Theorem 5.5).

Finally, I offer my gratitude to Thomas Brylawski (1944–2007) for many fruitful discussions on this topic. This approach to duality is due to him, and he proved many of the results given here. His influence on this author goes well beyond the present work, and this paper is dedicated to his memory. A memorial volume of the *European Journal of Combinatorics* includes a tribute to Tom and his work [7].

2 Definitions

2.1 Matroids via the Rank Function

There are many *cryptomorphically* equivalent ways to define a matroid. A standard reference is Oxley's textbook [12]. In this paper, we use the rank function.

Definition 2.1 A *matroid* M is a pair (S, r) where S is a finite set and $r : 2^S \to \mathbb{Z}^+ \cup \{0\}$ such that:

(R0) $r(\emptyset) = 0$ [*normalization*]
(R1) $r(A) \leq r(A \cup p) \leq r(A) + 1$ [*unit rank increase*]
(R2) $r(A \cap B) + r(A \cup B) \leq r(A) + r(B)$ [*semimodularity*]

Here, S is the *ground set* of the matroid. We will sometimes denote r by r_M if there is a need to distinguish between multiple rank functions. Assuming (R0) and (R1), we can replace (R2) with

Local semimodularity:

(R2′) If $r(A) = r(A \cup p_1) = r(A \cup p_2)$, then $r(A \cup \{p_1, p_2\}) = r(A)$.

When M is a matroid with ground set S, we can define independent sets, spanning sets and bases directly from the rank function:

- $I \subseteq S$ is *independent* if and only if $r(I) = |I|$.
- $T \subseteq S$ is *spanning* if and only if $r(T) = r(S)$.
- $B \subseteq S$ is a *basis* if and only if B is independent and spanning.

Thus, $B \subseteq S$ is a basis of the matroid M if $|B| = r(B) = r(S)$.

Three important operations motivated by graph theory can be defined for all matroids: duality, deletion, and contraction. These are usually defined in terms of independent sets or bases, but it is possible to define all three operations via the rank function.

Definition 2.2 Let $M = (S, r)$ be a matroid. Then define the dual matroid M^* as follows: $M^* = (S, r^*)$, where $r^*(A) = |A| + r(S - A) - r(S)$.

Using this definition, one can prove r^* satisfies (R0), (R1) and (R2), and so defines a matroid. One can also show that B is a basis for M^* if and only if $B = S - B'$, where B' is a basis for M, i.e., the bases for M^* are the complements of the bases of M.

Now we can define the matroid operations deletion and contraction.

Definition 2.3 Let $M = (S, r)$ be a matroid, and let $p \in S$.

1. *Deletion*: $M - p = (S - p, r')$, where $r'(A) = r(A)$ for any $A \subseteq S - p$.
2. *Contraction*: $M/p = (M^* - p)^*$.

Thus, both $M - p$ and M/p are matroids on the ground set $S - p$. It is then an elementary exercise to prove that $(M - p)^* = M^*/p$ and $(M/p)^* = M^* - p$.

2.2 Generalized Duality, Deletion and Contraction

Brylawski observed that the definitions of the dual matroid (Definition 2.2) and deletion and contraction (Definition 2.3) do not depend on the properties (R0), (R1) and (R2) that characterize the rank function of a matroid. This leads to the next definition, Brylawski's generalized duality, deletion and contraction. (In this paper, we will always assume that S is a finite set.)

Definition 2.4 Let $G = (S, r)$, where $r : 2^S \to \mathbb{Z}$ is a function satisfying $r(\emptyset) = 0$. Define duality, deletion and contraction:

- *Duality* $G^* = (S, r^*)$, where $r^*(A) := |A| + r(S - A) - r(S)$.
- *Deletion* $G - p = (S - p, r')$, where $r'(A) = r(A)$ for all $A \subseteq S - p$.
- *Contraction* $G/p := (G^* - p)^*$.

Table 1 Rank functions for rooted tree G of Fig. 1 and its generalized dual G^*

A	$\emptyset$	a	b	c	ab	ac	bc	abc
$r(A)$	0	1	0	1	2	2	1	3
$r^*(A)$	0	−1	0	0	0	−1	0	0

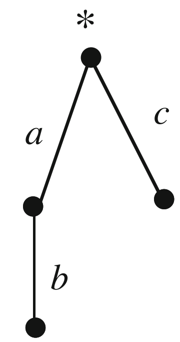

Fig. 1 Rooted tree G for Example 2.1

This definition of duality has the involution property: $(G^*)^* = G$. We omit the routine proof.

Proposition 2.1 *Let $G = (S, r)$, where $r : 2^S \to \mathbb{Z}$ is any function satisfying $r(\emptyset) = 0$. Then $(G^*)^* = G$.*

The next result is useful for computing the rank of a subset in G/p, and will also be needed in our proof of a deletion-contraction recursion for the Tutte polynomial (Theorem 3.1(1)). For clarity, we may denote the rank function of G by r_G.

Theorem 2.2 (Brylawski) *Let $G = (S, r)$, where $r : 2^S \to \mathbb{Z}$ satisfies $r(\emptyset) = 0$. Then, for all $p \in S$ and $A \subseteq S - p$, $r_{G/p}(A) = r(A \cup p) - r(p)$.*

Proof Let $A \subseteq S - p$. Then we compute as follows:

$$\begin{aligned} r_{G/p}(A) &= |A| + r_{G^*-p}((S-p)-A) - r_{G^*-p}(S-p) \\ &= |A| + r_{G^*}(S-p-A) - r_{G^*}(S-p) \\ &= |A| + \Big(|(S-p)-A| + r(S-(S-p-A)) - r(S)\Big) \\ &\quad - \Big(|S-p| + r(S-(S-p)) - r(S)\Big) \\ &= r(A \cup p) - r(p). \end{aligned}$$

Example 2.1 Let $S = \{a, b, c\}$ and define the rank function as in Table 1. This is the *branching greedoid* associated to the rooted tree of Fig. 1. (Sect. 4 gives more background information on greedoids.) Then the rank of a subset of edges A is the size of the largest rooted subtree contained in A. For instance, we have $r(\{b, c\}) = 1$ because c is the largest rooted subtree contained in $\{b, c\}$.

Then r is *not* the rank function of a matroid because, for example, $r(\{b, c\}) = 1$ while $r(S) = 3$, so the unit rank increase matroid property (R1) is violated. We use Definition 2.2 to find the rank for the dual G^*—see the last row of Table 1.

For instance, the dual rank $r^*(\{a, c\}) = |\{a, c\}| + r(b) - r(S) = -1$. This gives another way to see that r is not the rank function of a matroid, since, if it were, then the dual G^* would also be a matroid. But $r(A) < 0$ is impossible for matroids.

For the deletion $G - a$, we simply compute the rank function by restricting r to subsets avoiding a. The rank in the contraction can be computed via duality (as in Definition 2.3), or directly from Theorem 2.2. See Table 2.

For rooted graphs, one can check that these definitions of deletion and contraction via the rank function correspond to the usual graph-theoretic operations of deletion and contraction of edges.

Table 2 Rank functions of $G - a$ and G/a for the rooted tree of Fig. 1

A	$\emptyset$	b	c	bc
r_{G-a}	0	0	1	1
$r_{G/a}$	0	1	1	2

It is straightforward to generalize direct sums to arbitrary rank functions.

Definition 2.5 Let $G_1 = (S_1, r_1)$ and $G_2 = (S_2, r_2)$ where S_1 and S_2 are disjoint sets. For $A_i \subseteq S_i$, define $r(A_1 \cup A_2) = r_1(A_1) + r_2(A_2)$. Then $G_1 \oplus G_2 = (S_1 \cup S_2, r)$ is the *direct sum* of G_1 and G_2.

We omit the immediate proof of the next proposition.

Proposition 2.3 *Let $G_1 \oplus G_2$ be the direct sum of G_1 and G_2. Then*

$$(G_1 \oplus G_2)^* = G_1^* \oplus G_2^*.$$

3 The Tutte Polynomial

The Tutte polynomial is an important two-variable invariant for graphs and matroids. An extensive introduction to this polynomial can be found in [4]. We now extend the definition of the Tutte polynomial by using an arbitrary rank function $r : 2^S \to \mathbb{Z}$.

Definition 3.1 Let S be a finite set and let $r : 2^S \to \mathbb{Z}$ be any function. Then define $f(G; t, z)$ for $G = (S, r)$:

$$f(G; t, z) = \sum_{A \subseteq S} t^{r(S)-r(A)} z^{|A|-r(A)}.$$

Definition 3.1 generalizes the Tutte polynomial of a matroid to an arbitrary rank function. We do not assume any special properties for the function r. The exponent $r(S) - r(A)$ is the *corank* of A, and $|A| - r(A)$ is the *nullity* of A.

Example 3.1 Let $G = (S, r)$ for $S = \{a, b\}$ with the rank function $r : 2^S \to \mathbb{Z}$ given as follows: $r(\emptyset) = 3$, $r(a) = -1$, $r(b) = 7$ and $r(S) = 2$. Then

$$f(G; t, z) = \frac{1}{t^5 z^6} + \frac{1}{t z^3} + t + 1.$$

Thus, $f(G; t, z)$ need not be a polynomial.

We will generally assume $r(\emptyset) = 0$; this is needed to prove $G^{**} = G$ (Proposition 2.1). Note that $f(G; t, z)$ will be a polynomial precisely when the rank function satisfies the following two properties:

- $r(A) \le r(S)$ for all $A \subseteq S$ (*rank S maximum*), and
- $r(A) \le |A|$ for all $A \subseteq S$ (*subcardinality*).

Applying Definition 2.4 to this polynomial, we get the following.

Theorem 3.1 (Brylawski) *Let $r : 2^S \to \mathbb{Z}$ be any function satisfying $r(\emptyset) = 0$, and set $G = (S, r)$. Then*

1. *Deletion–contraction: For any $p \in S$,*

$$f(G; t, z) = t^{r(G)-r(G-p)} f(G - p; t, z) + z^{1-r(p)} f(G/p; t, z).$$

2. *Duality: Let G^* be the dual of G in the sense of Definition 2.4. Then*

$$f(G^*; t, z) = f(G; z, t).$$

Proof (1) We break up the subsets of S into two classes: Let $\mathcal{S}_1$ be the collection of all subsets of S containing p, and $\mathcal{S}_2$ be the collection of all subsets of S avoiding p.

Then

$$f(G;t,z) = \sum_{A\in\mathcal{S}_1} t^{r(S)-r(A)} z^{|A|-r(A)} + \sum_{A\in\mathcal{S}_2} t^{r(S)-r(A)} z^{|A|-r(A)}.$$

CASE 1: $A \in \mathcal{S}_1$. Then $r_{G/p}(S-p) = r(S) - r(p)$ and $r_{G/p}(A-p) = r(A) - r(p)$, so the corank of A (computed in G) equals the corank of $A - p$ (computed in G/p):

$$r(S) - r(A) = r_{G/p}(S-p) - r_{G/p}(A-p).$$

For the nullity, we have

$$|A| - r(A) = |A-p| + 1 - r_{G/p}(A-p) - r(p).$$

Thus

$$\begin{aligned}\sum_{A\in\mathcal{S}_1} t^{r(S)-r(A)} z^{|A|-r(A)} &= \sum_{A\in\mathcal{S}_1} t^{r_{G/p}(S-p)-r_{G/p}(A-p)} z^{(|A-p|-r_{G/p}(A-p))+(1-r(p))} \\ &= z^{1-r(p)} \sum_{B\subseteq S-p} t^{r_{G/p}(S-p)-r_{G/p}(B)} z^{|B|-r_{G/p}(B)} \\ &= z^{1-r(p)} f(G/p;t,z).\end{aligned}$$

CASE 2: $A \in \mathcal{S}_2$. Then $\sum_{A\in\mathcal{S}_2} t^{r(S)-r(A)} z^{|A|-r(A)} = t^{r(S)-r(S-p)} f(G-p;t,z)$. The proof is similar to case 1; we omit the details.

(2) Note that, for any $A \subseteq S$, we have $r^*(S) - r^*(A) = |S-A| - r(S-A)$ (since $r(\emptyset) = 0$) and $|A| - r^*(A) = r(S) - r(S-A)$. The result then follows from Definition 2.4(1).

The deletion-contraction recursion of Theorem 3.1(1) is a generalization of the greedoid version of this formula (Proposition 2.5 in [6]). In that formula, the $z^{1-r(p)}$ coefficient of the contraction term does not appear since $r(p) = 1$ for all points p that we contract. See Sect. 4 below.

Example 3.2 Returning to Example 2.1, we compute $f(G)$, $f(G-a)$ and $f(G/a)$:

A	$\emptyset$	a	b	c	ab	ac	bc	abc
$r(A)$	0	1	0	1	2	2	1	3
Term	t^3	t^2	t^3z	t^2	t	t	t^2z	1

Then

$$f(G;t,z) = (t+1)(t^2z + t^2 + t + 1),$$
$$f(G-a;t,z) = 1 + t + z + tz,$$
$$f(G/a;t,z) = (t+1)^2.$$

In this case, $r(a) = 1$ and $r(S) - r(S-a) = 2$, and the reader can verify $f(G;t,z) = f(G/a;t,z) + t^2 f(G-a;t,z)$, as required by Theorem 3.1(1).

If we delete and contract b, then we find $f(G-b;t,z) = (t+1)^2$ and $f(G/b;t,z) = t^3 + t^2 + \frac{t}{z} + \frac{1}{z}$ (and so $f(G/b)$ is not a polynomial). Now $r(b) = 0$ and $r(S) - r(S-b) = 1$, so 3.1 gives $f(G;t,z) = zf(G/b;t,z) + tf(G-b;t,z)$, which the reader can again verify.

For G^*, we find $f(G^*; t, z) = tz^3 + tz^2 + 1 + 2z + 2z^2 + z^3$.

Subset	∅	a	b	c	ab	ac	bc	abc
Dual rank	0	−1	0	0	0	−1	0	0
Dual term	1	tz^2	z	z	z^2	tz^3	z^2	z^3

Thus $f(G^*; t, z) = f(G; z, t)$, and we note that a term $t^m z^n$ of $f(G)$ corresponding to a subset A gives rise to the term $t^n z^m$ in $f(G^*)$ corresponding to the subset $S - A$.

We remark that when G is a matroid, then $r(G) = r(G-p)$ (provided p is not an isthmus) and $r(p) = 1$ (provided p is not a loop). Thus the recursion of Theorem 3.1(1) reduces to the familiar $f(G) = f(G - p) + f(G/p)$. It is also possible to interpret this recursion for isthmuses and loops; see [8] for one approach.

4 Greedoids

Greedoids are a generalization of matroids that were introduced in [9]. An extensive introduction appears in [1]. Although there are fewer axiomatizations of greedoids than there are of matroids, it is still possible define greedoids from a rank function.

Definition 4.1 A *greedoid* G is a pair (S, r) where S is a finite set and $r : 2^S \to \mathbb{Z}^+ \cup \{0\}$ such that:

(Gr0) $r(\emptyset) = 0$ [normalization]
(Gr1) $r(A) \le r(A \cup p)$ [increasing]
(Gr2) $r(A) \le |A|$ [subcardinality]
(Gr3) If $r(A) = r(A \cup p_1) = r(A \cup p_2)$, then $r(A \cup \{p_1, p_2\}) = r(A)$. [local semimodularity]

The rank function of a matroid satisfies these four properties, so matroids are greedoids. We remark that the greedoid normalization axiom (Gr0) is the same as the matroid axiom (R0), and the local semi-modularity for greedoids (Gr3) is identical to the matroid version (R2′).

We say A is a *feasible set* in the greedoid if $r(A) = |A|$. Feasible sets in greedoids play the same role as independent sets in matroids. *Bases* are defined to be maximal feasible sets, and, as with matroids, all bases have the same cardinality. One important difference between matroids and greedoids is that a greedoid is *not* uniquely determined by its collection of bases. In general, there may be many non-isomorphic greedoids with identical bases. A greedoid G is *full* if S is a basis of G, i.e., $r(G) = |S|$.

As an example, consider the rooted tree of Example 2.1. This is a greedoid; more generally, if G is a rooted graph with edges S, we get a greedoid with ground set S by defining the feasible sets to be the edges of the rooted subtrees of G. This is the *branching greedoid* associated to the rooted graph G.

Recall that duality, deletion and contraction are defined for greedoids from Definition 2.4. We can interpret greedoid deletion and contraction in terms of feasible sets.

Proposition 4.1 *Let G be a greedoid on the ground set S and let $p \in S$.*

1. *Deletion: $F \subseteq S - p$ is feasible in $G - p$ if and only if F is feasible in G.*
2. *Contraction: Assume $\{p\}$ is feasible. Then $F \subseteq S - p$ is feasible in G/p if and only if $F \cup p$ is feasible in G.*

We omit the straightforward proof. It is easy to see that $G - p$ is always a greedoid, but G/p is a greedoid only when $\{p\}$ is a feasible set or p is in no feasible sets:

Proposition 4.2 *Let $G = (S, r)$ be a greedoid and suppose $p \in S$ is in some feasible set F. Then G/p is a greedoid if and only if $\{p\}$ is feasible.*

Proof If $\{p\}$ is feasible, then $r_{G/p}(A) = r_G(A \cup p) - 1$. It is then a routine exercise to verify (Gr0) – (Gr3) for $r_{G/p}$. (Alternatively, we can use the feasible set characterization of Proposition 4.1(2).)

For the converse, we must show that G/p is not a greedoid when $\{p\}$ is not feasible. Let F be any feasible set containing p. Then, by Theorem 2.2, $r_{G/p}(F - p) = r_G(F) - r_G(p) = |F|$. But this violates the subcardinality property (Gr2) of greedoid rank functions.

If p is in no feasible sets in the greedoid G, we call p a *greedoid loop*. If p is a greedoid loop, then we may use Theorem 2.2 to verify $G/p = G - p$. We remark that this is not completely standard in greedoid theory. In particular, contraction is not usually defined for greedoid loops. But our rank function approach allows us to contract loops, preserving a greedoid structure. (Of course, we can also contract non-feasible, non-loop elements of S, but G/p may not be a greedoid; see Example 3.2.)

We can formulate dual versions of the greedoid rank axioms (Gr0)–(Gr3). This gives us a direct characterization of duality for these structures.

Theorem 4.3 (Brylawski) *Let r be the rank function for a greedoid $G = (S, r)$. Then, for all $B \subseteq S$, the dual rank r^* satisfies the following:*

(*Gr0**) $r^*(\emptyset) = 0$ [*normalization*]
(*Gr1**) $r^*(B \cup p) \leq r^*(B) + 1$ [*unit rank increase*]
(*Gr2**) $r^*(B) \leq r^*(S)$ [*rank S maximum*]
(*Gr3**) *If* $r^*(B - p) = r^*(B - q) = r^*(B) - 1$, *then* $r^*(B - \{p, q\}) = r^*(B) - 2$. [*local rank decrease*]

Proof We omit the straightforward proofs of (Gr0*) and (Gr2*). For (Gr1*), assume $p \notin B$ and set $A = S - (B \cup p)$, so $A \cup p = S - B$.

Then

$$
\begin{aligned}
r^*(B \cup p) &= |B \cup p| + r(S - (B \cup p)) - r(S) \\
&= |B| + 1 + r(A) - r(S) \\
&\leq |B| + 1 + r(A \cup p) - r(S) \text{ (by (Gr1))} \\
&= |B| + 1 + r(S - B) - r(S) \\
&= r^*(B) + 1.
\end{aligned}
$$

For (Gr3*), set $A = S - B$. Then $r^*(B) = r^*(B - p) + 1$ implies $r(A) = r(A \cup p)$, and, similarly, $r(A) = r(A \cup q)$. By (Gr3), we then get $r(A \cup \{p, q\}) = r(A)$. Then

$$
\begin{aligned}
r^*(B - \{p, q\}) &= |B - \{p, q\}| + r(A \cup \{p, q\}) - r(S) \\
&= |B| - 2 + r(A) - r(S) \\
&= |B| + r(S - B) - r(S) - 2 \\
&= r^*(B) - 2.
\end{aligned}
$$

In Example 2.1, the rooted tree is a greedoid, but its dual is not (in particular, the rank function r^* for G^* was negative for some subsets). When is the dual of a greedoid also a greedoid? The answer leads us back to matroids.

Theorem 4.4 (Brylawski) *Let $\mathcal{M}$ be the class of all matroids, $\mathcal{G}$ the class of all greedoids and $\mathcal{G}^* = \{G^* : G \in \mathcal{G}\}$. Then*

$$\mathcal{G} \cap \mathcal{G}^* = \mathcal{M}.$$

Proof Matroids are greedoids, so $\mathcal{M} \subseteq \mathcal{G}$. Taking the duals gives $\mathcal{M}^* \subseteq \mathcal{G}^*$. Since duals of matroids are matroids, we also have $\mathcal{M}^* = \mathcal{M}$, so $\mathcal{M} \subseteq \mathcal{G} \cap \mathcal{G}^*$.

For the converse, note that if $G \in \mathcal{G} \cap \mathcal{G}^*$, then the rank function r for G satisfies (Gr0), (Gr1*) and (Gr3). But these three properties characterize the rank function of a matroid, so G is a matroid, i.e., $\mathcal{G} \cap \mathcal{G}^* \subseteq \mathcal{M}$.

When G is a rooted graph, recall that a subset of edges A is a feasible set in the branching greedoid if the edges of A form a rooted tree. Example 2.1 shows the dual G^* is not generally a greedoid. But we can determine precisely when $r^*(A) \geq 0$ for all $A \subseteq S$ in this case.

Proposition 4.5 *Let G be a connected simple rooted graph with edges S and branching greedoid rank function r. Then $r^*(A) \geq 0$ for all $A \subseteq S$ if and only if every vertex of G is adjacent to the root.*

Proof Let $V = \{v_0, v_1, v_2, \ldots, v_n\}$ be the collection of vertices of G, with root vertex v_0, and suppose v_0 is adjacent to each v_i for $1 \leq i \leq n$. Write e_i for the edge joining vertices v_0 and v_i. Let $A \subseteq S$ and suppose $e_i \in A$ for $1 \leq i \leq k$ and $e_j \notin A$ for $k+1 \leq j \leq n$ (for some k). Then $r(S) = n$, $|A| \geq k$ and $r(S - A) \geq n - k$, so

$$r^*(A) = |A| + r(S - A) - r(S) \geq k + (n - k) - n \geq 0.$$

For the converse, suppose the root vertex v_0 is adjacent to vertices $v_1, v_2, \ldots, v_n$, but there is some vertex u that is not adjacent to v_0. Let e_i be the edge joining v_0 and v_i (as above), and set $A = \{e_1, e_2, \ldots, e_n\}$. Then $|A| = n$ and $r(S - A) = 0$. Since G is connected and has at least $n + 1$ non-root vertices, we must have $r(S) > n$. Thus, $r^*(A) = n + 0 - r(S) < 0$. □

5 Antimatroids, Demi-matroids and Duality

5.1 Antimatroids

Antimatroids are an important class of greedoids that have been rediscovered many times in the literature; they originally appeared in [5] in 1940. An interesting account of the history of the different formulations and discoveries of antimatroids appears in [11]. For our purposes, an antimatroid is a greedoid in which the union of feasible sets is always feasible.

Definition 5.1 Let $G = (S, r)$ be a greedoid with rank function r and feasible sets $\mathcal{F}$. Then G is an *antimatroid* if $F_1 \cup F_2 \in \mathcal{F}$ whenever $F_1, F_2 \in \mathcal{F}$.

There are several important combinatorial structures that admit an antimatroid structure in a natural way: a partial list includes trees, rooted trees, rooted directed trees, finite subsets of Euclidean space, posets (in three different ways), and vertices in chordal graphs. If the antimatroid has no greedoid loops, then the antimatroid is a full greedoid, i.e., S is a feasible set. All of the antimatroids listed above are full.

We are interested in interpreting the dual rank function of an antimatroid directly from the antimatroid structure. One obvious problem is the presence of subsets with negative rank. In fact, in Example 2.1, the dual rank $r^*(A) \leq 0$ for all $A \subseteq S$ (see Table 1). This is true generally when the greedoid is full.

Proposition 5.1 *Let $G = (S, r)$ be a full greedoid. Then $r^*(A) \leq 0$ for all $A \subseteq S$.*

Proof G is a full greedoid if and only if $r(S) = |S|$. Then $r^*(A) = |A| + r(S - A) - r(S) = r(S - A) - |S - A| \leq 0$ by the subcardinality property of greedoid rank functions (Gr2). □

When G is an antimatroid with S feasible, we say a subset $C \subseteq S$ is *convex* if $S - C$ is feasible. The convex sets provide a complementary way to study antimatroids, and arise naturally in a variety of combinatorial structures. See Example 5.1.

Definition 5.2 Let (S, r) be an antimatroid and let $A \subseteq S$. The *convex closure* $\overline{A}$ is the smallest convex set that contains A.

Equivalently, $\overline{A}$ is the intersection of all convex sets containing A.

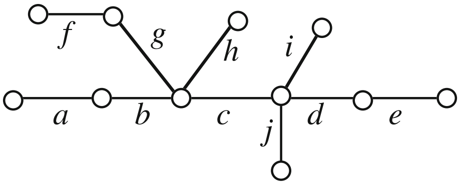

Fig. 2 Tree for Example 5.1

Example 5.1 Let T be the (non-rooted) tree in Fig. 2. Let S be the set of edges of the tree, and define $A \subseteq S$ to be feasible if $S - A$ is a subtree. Then this gives an antimatroid structure, where the rank function $r(A)$ is the size of the largest feasible subset of A. Then C is convex if and only if the edges of C form a subtree.

This is the *pruning* antimatroid associated to the tree. For instance, the subset $A = \{a, d, e, f\}$ is feasible since its complement $S - A = \{b, c, g, h, i, j\}$ is a subtree. Thus, $S - A$ is convex. Equivalently, A is feasible since the edges of A can be *pruned* from the tree by repeatedly removing leaves. Since A is feasible, we have $r(A) = 4$, and $r^*(S - A) = |S - A| + r(A) - r(S) = 0$.

For $A = \{b, e, h\}$, we find A is not feasible. Then $r(A) = 2$ since the subset $\{e, h\}$ is the largest feasible subset of A. We also see that A is not convex. The smallest subtree containing A is $\{b, c, d, e, h\}$, so $\overline{A} = \{b, c, d, e, h\}$.

We now give a combinatorial interpretation for the dual rank in an antimatroid.

Theorem 5.2 *Let $G = (S, r)$ be a full antimatroid. Then $r^*(A) = -|\overline{A} - A|$, where $\overline{A}$ is the convex closure of A.*

Proof We first show $r(S - A) = r(S - \overline{A})$. Now $S - \overline{A} \subseteq S - A$ gives $r(S - \overline{A}) \leq r(S - A)$ (from greedoid rank property (Gr1)). But $\overline{A}$ is convex, so $S - \overline{A}$ is feasible. Thus, $r(S - \overline{A}) = |S - \overline{A}|$.

Now suppose $r(S - \overline{A}) < r(S - A)$. Then there is a feasible set $F \subseteq S - A$ with $|F| > |S - \overline{A}|$. Then $F \cup (S - \overline{A})$ is a feasible set that properly contains $S - \overline{A}$, so the complement $S - (F \cup (S - \overline{A}))$ is a convex set containing A that is properly contained in $\overline{A}$. Thus, $\overline{A}$ is not the smallest convex set containing A, a contradiction. We conclude $r(S - \overline{A}) = r(S - A)$.

Now

$$\begin{aligned} r^*(A) &= |A| + r(S - A) - r(S) \\ &= |A| + r(S - \overline{A}) - r(S) \\ &= |A| + |S - \overline{A}| - |S| \\ &= -|\overline{A} - A|. \end{aligned}$$

□

For example, let $A = \{a, d, f\}$ in the tree in Fig. 2. Then $r(S - A) = 4$ since $\{e, h, i, j\}$ is the largest feasible subset of $S - A$, so $r^*(A) = |A| + r(S - A) - r(S) = 3 + 4 - 10 = -3$.

Now $\overline{A}$ is the smallest subtree containing A, so $\overline{A} = \{a, b, c, d, f, g\}$. Thus $\overline{A} - A = \{b, c, g\}$, and $|\overline{A} - A| = 3$, as required by Theorem 5.2.

An immediate corollary of Theorem 5.2 is the following characterization of convex sets in an antimatroid in terms of the dual rank r^*.

Corollary 5.3 *Let $G = (S, r)$ be a full antimatroid. Then C is convex if and only if $r^*(C) = 0$.*

5.2 Demi-matroids

Demi-matroids were introduced in [3], where they provide a more general setting for Wei's duality theorem for codes [13].

Definition 5.3 A *demi-matroid* is a triple (S, r, s) with S a finite set and rank functions $r, s : 2^S \to \mathbb{Z}^+ \cup \{0\}$ satisfying

1. $r(A) \le |A|$ and $s(A) \le |A|$
2. If $A \subseteq B$, then $r(A) \le r(B)$ and $s(A) \le s(B)$.
3. $|S - A| - r(S - A) = s(S) - s(A)$.

It follows immediately from this definition that r and s also satisfy the complementary version of (3):

$|S - A| - s(S - A) = r(S) - r(A)$.

If M is a matroid with rank function r, then the function s is simply the dual rank r^*. Thus, matroids are demi-matroids, where $s = r^*$. However, if S is a finite set with arbitrary rank function r, then the generalized dual rank function r^* of Definition 2.4 need not satisfy the properties required of s.

For instance, consider the greedoid of Example 2.1. Then $r(A)$ is a non-negative integer for any subset A, and the greedoid rank function satisfies properties (1) and (2) of the demi-matroid properties (Definition 5.3). Further, r and the dual rank function r^* satisfy (3):

$|S - A| - r(S - A) = r^*(S) - r^*(A)$.

However, the dual rank function r^* does not satisfy (2) and $r^*(A) < 0$ for $A = \{a\}$. (Definition 5.3(2) is violated for $A = \{c\}$ and $B = \{a, c\}$.)

We are interested in characterizing demi-matroids via the rank function r. We will need the following lemma, whose straightforward inductive proof is omitted.

Lemma 5.4 *Let S be a finite set with rank function $r : 2^S \to \mathbb{Z}^+ \cup \{0\}$ satisfying the subcardinality property: $r(A) \le r(B)$ whenever $A \subseteq B$. Then the following two properties are equivalent:*

(R1) $r(A) \le r(A \cup p) \le r(A) + 1$ [*unit rank increase*]
(MN) If $A \subseteq B$, then $|A| - r(A) \le |B| - r(B)$. [*monotone nullity*]

We also point out that monotone nullity (propertry (MN)) can be expressed in other ways. For instance, it is immediate that this property is equivalent to

$A \subseteq B \Rightarrow r(B) - r(A) \le |B - A|$.

Theorem 5.5 *Let S be a finite set with rank function $r : 2^S \to \mathbb{Z}$. Then the triple (S, r, r^*) is a demi-matroid if and only if r satisfies, for all $p \in S$ and $A, B \subseteq S$:*

(a) $0 \le r(A) \le |A|$, [*nonnegative, subcardinality*]
(b) if $A \subseteq B$, then $r(A) \le r(B)$ [*monotone rank*]
(c) $r(A \cup p) \le r(A) + 1$ [*unit rank increase*]

Proof Suppose r satisfies the three conditions (a), (b) and (c). We show the triple (S, r, r^*) is a demi-matroid. From Definition 2.4, we have $r^*(A) = r(S - A) + |A| - r(S)$. This immediately implies that (3) holds in Definition 5.3. Thus, to show (S, r, r^*) is a demi-matroid, we must show the dual rank r^* also satisfies (for all subsets $A, B \subseteq S$):

1. $0 \le r^*(A) \le |A|$, and
2. if $A \subseteq B$, then $r^*(A) \le r^*(B)$.

First, we show $0 \le r^*(A)$ for all $A \subseteq S$. But $0 \le r^*(A)$ if and only if $r(S) \le r(S - A) + |A|$. Since $S - A \subseteq S$, we have $|S - A| - r(S - A) \le |S| - r(S)$ (by Lemma 5.4), and so $r(S) \le r(S - A) + |A|$.

To show $r^*(A) \le |A|$, we note this is equivalent to $r(S - A) + |A| - r(S) \le |A|$, i.e., $r(S - A) \le r(S)$. This now follows directly from condition (b).

It remains to show that if $A \subseteq B$, then $r^*(A) \leq r^*(B)$. Now

$$\begin{aligned} r^*(A) \leq r^*(B) &\Leftrightarrow r(S-A)+|A|-r(S) \leq r(S-B)+|B|-r(S) \\ &\Leftrightarrow r(S-A)-r(S-B) \leq |S-A|-|S-B| \\ &\Leftrightarrow |B'|-r(B') \leq |A'|-r(A') \end{aligned}$$

where $A' = S - A$ and $B' = S - B$, with $B' \subseteq A'$. This now follows from monotone nullity (Lemma 5.4).

For the converse, we first observe that if (S, r, s) is a demi-matroid, then (3) in Definition 5.3 forces $s = r^*$, where $r^*(A) = r(S-A)+|A|-r(S)$. Then r must satisfy conditions (a) and (b) (this follows from Def 5.3(1) and (2)).

It remains to show r also satisfies the monotone nullity property (MN) (Lemma 5.4). Assume $A \subseteq B$. Then the argument given above shows that r satisfies (MN) if and only if r^* satisfies Def 5.3(2):

$$|A|-r(A) \leq |B|-r(B) \Leftrightarrow r^*(S-B) \leq r^*(S-A).$$

Since $S-B \subseteq S-A$ and (S, r, r^*) is a demi-matroid, we know $r^*(S-B) \leq r^*(S-A)$. This completes the proof. □

In Example 2.1, the greedoid rank function is nonnegative and subcardinal, so it satisfies condition (a) of Theorem 5.5. Further, r satisfies the monotone rank property (condition (b)). But r does not satisfy the unit rank increase property (R1). (Set $A = \{b\}$ and $B = \{a, b\}$, for instance.) This confirms the fact that the greedoid of Example 2.1 is not a demi-matroid. (Of course, r does not satisfy the monotone nullity property, either, which is violated for the same A and B).

Lemma 1 of [3] shows that the rank function r of a demi-matroid satisfies the unit rank increase property (R1). We also note that the three properties (a), (b) and (c) of Theorem 5.5 are not sufficient to define a matroid. For instance, Example 2 of [3] has $S = \{a, b\}$, with $r(A) = 0$ for $A = \emptyset, \{a\}$ or $\{b\}$, and $r(S) = 1$. Then $r^* = r$, and (S, r, r^*) is a demi-matroid, so r and r^* satisfy (a), (b) and (c) from Theorem 5.5. But this is not a matroid; the semimodular property (R2) is violated.

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