Orthogonal Latin Garage Doors

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The House

When we moved into our new house in the fall of 1986, the roof was gray, the siding was gray and the two garage doors were white. In addition, the weather in Easton, Pennsylvania, is frequently gray, at least during the winter. The four of us (two mathematicians and our two daughters) came to the same conclusion: the exterior of our house needed something colorful to cheer up the neighborhood.

The Project

The two wooden garage doors are constructed in horizontal sections, with four sections making up a door. Each section has four raised rectangular panels, so each door has 16 panels arranged in a 4 x 4 grid. The doors need to be painted every few years, so we soon hit on an obvious solution to our gray lives: Paint the doors some color that wasn't white.

We talked about the particulars of such a painting project and came up with a few constraints:

1. Do something mathematical. Since both adults in the house are mathematicians and we frequently have students over for various activities, this seemed important.
2. Don't do something that would completely alienate us from our neighbors. We were reasonably sure painting rectangles on the doors solid colors would keep us within the bounds of 'semi-normal neighborhood behavior.'
3. Do something colorful. This was the motivation for the project to begin with.
4. Choose colors that we could live with for a long time. We also needed the colors to be easy to distinguish.

The Math

We decided to paint the panels of the doors to make a pair of 4 x 4 orthogonal Latin squares. A *Latin square* is an n x n array, where each member of the array is one of the numbers from 1 to n, so that each number between 1 and n appears exactly once in each row and each column. Two n x n Latin squares are *orthogonal* if, when you superimpose one array on the other to obtain ordered pairs, each one of the n^2 possible ordered pairs appears precisely once. (To see these ordered pairs clearly, superimpose one array on the other and then slide one of the arrays a bit to the left.)

Latin squares are important in all sorts of applications in experiment design, geometry and graph theory, and they arise naturally as group multiplication tables. The subject of Latin squares is more than 200 years old; amateurs attempting to design whist tournaments did much of the early development. Almost any text on combinatorics treats this topic; in fact, a pair of orthogonal 4 x 4 Latin squares decorates the cover of van Lint and Wilson's interesting text *A Course in Combinatorics*. It is possible to find a pair of n x n orthogonal Latin squares for every n except 2 and 6. Euler conjectured that no orthogonal Latin squares exist when n is congruent to 2 *mod* 4, but in 1959, this conjecture was disproved when a pair of 10 x 10 orthogonal Latin squares was found at the University of North Carolina by Parker, Bose and Shrikhande. A nice, short introduction to the topic can be found at http://buzzard.ups.edu/squares.html. The field remains an active area of research, with many interesting open problems waiting for new approaches.

For our purposes, we needed a pair of orthogonal 4 x 4 Latin squares. Algorithms exist for finding orthogonal pairs of Latin squares using projective planes, but, because we wanted one of the doors to have nice symmetry, we used trial and error. We first tried the square below, but unfortunately, we showed that no other Latin square is orthogonal to it.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Then we tried a more symmetric square, pictured on the left below. Trial and error produced a square orthogonal to it. (Although it is always possible to relabel the squares so that the first row in each square is 1, 2, 3, 4, we liked the two squares given below better.)

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

It is not hard to show that there can be at most three pairwise orthogonal 4 x 4 Latin squares, and, in general, at most n-1 pairwise orthogonal n x n Latin squares. This maximum number is achieved whenever there is a projective plane of order n. All known projective planes have order $n = p^k$ for some prime p and some integer $k \geq 1$, so these are the only values of n where the full complement of orthogonal squares is known to exist. Since we only have a two-car garage, our work was done. Is there a third Latin square that is orthogonal to the two given above? Find it!

The Paint

We were now ready to plan the painting. Rebecca and Hannah (our daughters) felt very strongly about the colors they liked and the color combinations they would not allow. This restriction banished any oranges or yellows. Also, the colors had to look good together, and not be too different (this restriction banned blue and red as two of the colors). Rebecca's favorite color was teal and purple was Hannah's favorite, so those two colors were in. Then we took a short trip to our local hardware store and made final choices for the remaining colors, staying on the blue side of the spectrum. Our helpful salesman mixed the paints after feigning interest in a short lecture in combinatorics.

When the time came to paint, each of us was responsible for painting their favorite color.
The correspondence between number and color (and painter) was

1. Purple (Hannah);
2. Light Blue (Liz, who calls it Carolina blue);
3. Dark Blue (Gary);
4. Teal (Rebecca).

The painting took two days, including a primer and two coats for most of the colors. We applied masking tape to each bounding edge of each rectangle (a non-trivial task, as there are 128 such edges on the two doors). The hardest part of the project was applying and removing this tape from the doors; Rebecca and Hannah were less interested in this part.

The Reactions

We love our doors! More fun has been the reaction of other people, which has been almost uniformly positive. (On the other hand, would you approach a stranger and tell them their garage doors are hideous?)

The reactions to the project began while we were painting. One neighbor stopped by and asked 'What are you doing?' We gave a fairly short explanation, and he seemed satisfied and allowed that it seemed a good family project. A passing driver slowed down long enough to shout 'My daughter thinks it's the coolest!' and a British neighbor said 'Actually, it's rather neat,' (read this with a British accent). An artist (who was selling us a car) got out his sketchbook to write the patterns down.

People understand the Latin square property quickly; it's easy to see that each color appears once in each row and column on each door. It's harder to explain orthogonality; we tell them to focus on the positions of the purple rectangles (for example) on one door and to find the corresponding positions on the other door. They can see that all four colors appear in these corresponding positions on the second door. Then they can check that this property holds for the other three colors, if they want, or they can take our word for it.

Most people who study the doors find the symmetry on the left door. There is also symmetry on the right door, but it's harder to see. Sarah McMahon (a historian at

Bowdoin College, and the sister and sister-in-law of the authors) realized that the positions of each individual color on the right door are the same up to rotation, reflection or both rotation and reflection. She was very excited when she discovered that pattern.

Teenagers like the doors; most comments are on the order of ‘They’re really cool!’ or ‘I think they’re funky.’ Our favorite reaction was last Halloween, though. A trick-or-treating teen said ‘Your garage doors are really nice. I heard there is a mathematical pattern to them.’ Maybe next year we’ll pass out reprints instead of candy.

The Moral

Neighbors are surprisingly tolerant of mathematical home improvement. Without painting numbers, equations or theorems, we helped spice up the street and we also made it much easier to give directions to our house (‘Look for the colorful garage doors.’). One of us offers a prize for anyone (excluding mathematicians) who can figure out the pattern; no one has claimed it yet. We are certainly not the only mathematicians who have decorated their house in a mathematical theme, but we are fairly certain that we are the only ones on our block.

There are other possible themes we (or you) could explore. We could mow crop circles in our lawn, or paint the Fano configuration on the front door. Finally, all of us can all find inspiration in the ideas of the mathematician Carl Frederick Gauss. The great Gauss once suggested the following enormous project: By cutting down a massive number of trees in Siberia, he hoped to give the largest proof ever of the Pythagorean Theorem. The goal of such a project was also to impress the neighbors, but on a grand scale. Presumably, such a figure would be visible to extraterrestrials, who would then assume that intelligent life existed on Earth. We have no plans to undertake this project.