DISTINGUISHED VERTICES IN PROBABILISTIC ROOTED GRAPHS

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Abstract. The expected number of vertices that remain joined to the root vertex s of a rooted graph G_s when edges are prone to fail is a polynomial $EV(G_s; p)$ in the edge probability p that depends on the location of s. We show that optimal locations for the root can vary arbitrarily as p varies from 0 to 1 by constructing a graph in which every permutation of k specified vertices is *the* 'optimal' ordering for some $p, 0 < p < 1$. We also investigate zeroes of $EV(G_s; p)$, proving that the number of vertices of G is bounded by the size of the largest rational zero.

1. Introduction

When the edges in a rooted graph fail, the number of vertices adjacent to the root may be diminished, and this may have important consequences for repair of the network. We are interested in the expected number of vertices which remain connected to the root vertex in a probabilistic network. We assume each edge of the network independently operates with the same probability p, and the vertices of the network can not fail. These somewhat unrealistic assumptions are the standard ones of network reliability, and have surprising applications and connections with other areas of mathematics and statistical physics. See [8] for an extensive introduction to network reliability and the reliability polynomial of a graph.

In this work, we are interested in the situation when one vertex of the graph is distinguished as a root vertex. Then the expected number of vertices reachable from the root will be a *polynomial* in p that we denote $EV(G; p)$, and this polynomial may be taken as a measure of the reliability of the rooted network. A related invariant has been studied by Colbourn [9] (called *network resilience*) and Siegrist [17] and also by Siegrist, Amin and Slater [2, 3, 18] (called *pair connected reliability*). More motivation and background can be found in [4]. This paper continues the study initiated in [1] and [4].

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For a given graph, a fundamental problem concerns the optimal location for the root vertex. Let G be a graph and let s be a vertex of G, and let G_s denote the rooted graph obtained by rooting G at s. For each vertex s in the graph, we can compute $EV(G_s;p)$ to determine the best location for the root. We generally expect the location of the optimal vertex to depend on the value of p. For example, if p is close to 0, then the optimal location for the root is simply the vertex of highest degree (when there is a unique such vertex); when p is close to 1, we expect most edges of the graph to survive, so a more centrally located vertex is a better choice for the root.

How often can the optimal vertex change as p varies from 0 to 1? This question is our primary motivation:

> **Problem**: Construct a graph G with adjacent vertices v and w such that the graphs of the polynomials $EV(G_v;p)$ and $EV(G_w;p)$ cross more than once as p varies from 0 to 1.

We give such a construction in Proposition 2.5. Using completely elementary techniques, we construct a tree T in which the optimal location for the root vertex changes arbitrarily often between two adjacent vertices as p varies from 0 to 1.

The problem of reliability polynomials *crossing* has been studied before. The *all-terminal reliability* $R(G)$ of a graph G is the probability that the graph is connected when each edge is independently subject to failure. This well-studied invariant [8] is also a polynomial in p. Kelmans [14] constructs examples of pairs of graphs G_1 and G_2 with the same number of vertices and edges having the property that $R(G_1) - R(G_2)$ can have an arbitrary (finite) number of roots between 0 and 1, so neither G_1 nor G_2 can be considered 'better.' Kelmans also proves similar results for *two-terminal* reliability, another polynomial that gives the probability that two distinguished vertices remain connected. Similar examples concerning the non-existance of uniformly optimal graphs appear in [13] and [15].

In the other direction, *optimal graphs* do exist in certain circumstances. For pair-connected reliability, Siegrist, Amin and Slater [18] determine the optimal unicyclic graphs. We emphasize the difference between these approaches and ours: Rather than determine an optimal graph with given parameters (n vertices and m edges), we assume the graph G is given and determine the potential optimal roots among the vertices of G.

The construction used in Proposition 2.5 can be generalized so that the optimal location for the root can vary among $k > 2$ vertices as p varies from 0 to 1. Using a slightly more complicated argument (but elementary techniques, as before), we construct a graph G on n vertices with k distinguished vertices $S = \{v_1, \ldots, v_k\}$ so that, for any permutation $\pi \in S_k$, we have, for any $u \notin S$ and some $0 < p < 1$,

$$EV(G_{v_{\pi(1)}}; p) > EV(G_{v_{\pi(2)}}; p) > \cdots > EV(G_{v_{\pi(k)}}; p) > EV(G_u; p).$$

In Section 3, we consider a combinatorial interpretation of $EV(G_s; p)$. In particular, we find bounds on the zeroes of $EV(G_s; p)$ in terms of n, the number of vertices. Our main result in this direction is Proposition 3.1, namely, the number of vertices is bounded by the size of the largest rational root, and this bound is sharp. This connection is a bit mysterious to us; $EV(G_s; p)$ has no immediate physical interpretation in terms of probability for values of $p > 1$. In addition, we construct rooted graphs with polynomials having specified positive integer zeroes (Proposition 3.4) and specified negative rational zeroes (Proposition 3.2).

We point out that the proofs of all of these results are routine; the only tools we need are standard techniques for manipulating polynomials. Furthermore, many of our constructions are based on trees, and $EV(T_s; p)$ is especially simple in this case. The fact that so much freedom is allowed for vertex location in these cases indicates how difficult the general problem of vertex location is. This is explored in some detail for other families of graphs in [1, 4, 10, 11].

2. Optimal root vertex location

Let G be a graph with root s, let E denote the edges of G and suppose each edge of the graph has the same independent probability p of operating. For a subset of edges $S \subseteq E$, we let $r(S)$ denote the number of vertices (besides the root) in the component of the subgraph S that contains the root. ($r(S)$ is the *pruning rank* of the *branching greedoid* associated to G_s.)

Then the expected rank of a subset of edges is given by

$$EV(G_s; p) = \sum_{S \subseteq E} r(S) p^{|S|} (1-p)^{|E-S|}. \tag{1}$$

For a vertex $v \in V$, let $Pr(v)$ denote the probability that v remains connected to the root vertex s. The following expression follows immediately from the additivity of expected value:

$$EV(G_s; p) = \sum_{v \in V - \{s\}} Pr(v). \tag{2}$$

This gives a very easy way to interpret $EV(T_s; p)$ when T is a tree.

Proposition 2.1. *[Corollary 2.6 of [1]] Let T_s be a tree rooted at s, and let a_i denote the number of vertices at distance exactly i from s. Then*

$$EV(T_s;p) = \sum_{i>0} a_i p^i.$$

For a graph G with vertices u and v and a fixed p between 0 and 1, we write $u >_p v$ if $EV(G_u;p) > EV(G_v;p)$. For all but a finite number of values of p, this induces a total ordering of the vertices of G. The next example motivates the main result of this section.

Example 2.2. Let G be the graph of Figure 1. Then we have

$$\begin{aligned}
EV(G_1;p) &= 5p + 5p^2 + 7p^3 + 3p^4 - 6p^5 \\
EV(G_2;p) &= 4p + 8p^2 + 6p^3 - 4p^5 \\
EV(G_3;p) &= 3p + 11p^2 + 5p^3 - 3p^4 - 2p^5 \\
EV(G_4;p) &= 5p + 2p^2 + 7p^3 + 5p^4 - 3p^5 - 2p^6
\end{aligned}$$

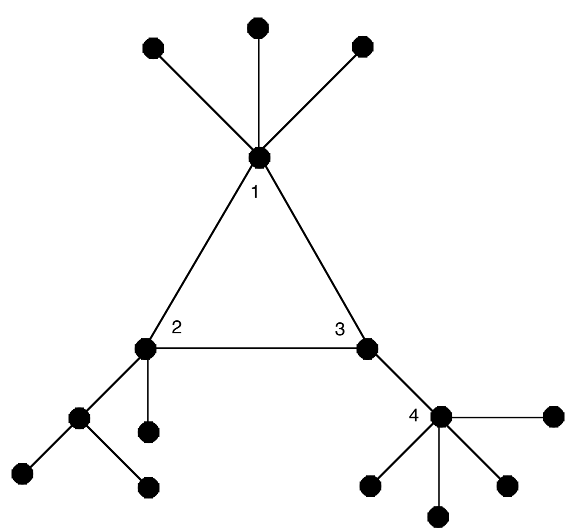

FIGURE 1. Graph illustrating vertex ordering $>_p$.

In Table 1, we list the ordering of the vertices 1 – 4 for various values of p.

p range	$>_p$ Order
$0 \quad < p < .1765\ldots$	$1 >_p 4 >_p 2 >_p 3$
$.1765\ldots \quad < p < .2491\ldots$	$1 >_p 2 >_p 4 >_p 3$
$.2491\ldots \quad < p < .5$	$1 >_p 2 >_p 3 >_p 4$
$.5 \quad < p < 1$	$3 >_p 2 >_p 1 >_p 4$

Table 1. Optimal vertex ordering for graph of Fig. 1.

We are interested in creating graphs in which the induced order $>_p$ will achieve any permutation of a specified collection of k vertices for some value of p between 0 and 1. When $k = 2$, we can accomplish this using trees.

We will need the following lemma that follows immediately from Proposition 2.1.

Lemma 2.3. *Let $f(p) = \sum_{k=1}^{n} a_k p^k$, where each a_k is a positive integer. Then there is a rooted tree T_s with $EV(T_s; p) = f(p)$.*

The next lemma shows how to construct a tree in which adjacent vertices can swap arbitrarily often as p varies from 0 to 1.

Lemma 2.4. *Let $h(p)$ be any polynomial with integer coefficients such that $h(0) = h(1) = 0$. Then there is a tree with adjacent vertices u and w such that $EV(T_u; p) - EV(T_w; p) = h(p)$.*

Proof. Let (u, w) be an edge of T and let T' and T'' denote the two subtrees that remain when (u, w) is deleted. Then we express $EV(T_u; p)$ and $EV(T_w; p)$ in terms of the polynomials on the rooted subtrees T'_u and T''_w:

$$EV(T_u; p) = EV(T'_u; p) + pEV(T''_w; p) + p$$

and

$$EV(T_w; p) = EV(T''_w; p) + pEV(T'_u; p) + p.$$

Thus $EV(T_u; p) - EV(T_w; p) = (1 - p)\left(EV(T'_u; p) - EV(T''_w; p)\right).$

Write $h(p) = (1 - p)\sum_{k=1}^{n} c_k p^k$ (this is possible since $h(0) = h(1) = 0$). We must show there are positive integers a_k, b_k and rooted trees T'_u and T''_w such that $EV(T'_u; p) = \sum_{k=1}^{n} a_k p^k, EV(T''_w; p) = \sum_{k=1}^{n} b_k p^k$, and $a_k - b_k = c_k$ for $1 \le k \le n$.

But this is immediate: for instance, if $c_k > 0$, then choose $a_k = c_k + 1$ and $b_k = 1$, if $c_k = 0$, choose $a_k = b_k = 1$, and if $c_k < 0$, choose $a_k = 1$ and $b_k = 1 - c_k$. (Obviously, there are many other solutions.)

□

We can now prove our main result for trees.

Proposition 2.5. *Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_t = 1$, where each α_i is rational. Then there is a tree T with adjacent vertices u and w such that the optimal location for the root is u if $\alpha_{2k} < p < \alpha_{2k+1}$ (for $0 \leq k \leq \lfloor \frac{t-1}{2} \rfloor$) and w if $\alpha_{2k-1} < p < \alpha_{2k}$ (for $1 \leq k \leq \lfloor \frac{t}{2} \rfloor$).*

Proof. Let $h(p) = N(1-p)p\prod_{i=1}^{n}(\alpha_i - p)$, where N is the product of the denominators appearing in the α_i. Then $h(p)$ has integer coefficients and $h(0) = h(1) = 0$, so it satisfies the hypothesis of Lemma 2.4. Thus, there is a tree T and a pair of adjacent vertices u and w with $EV(T_u;p) - EV(T_w;p) = h(p)$.

Now $EV(T_u;p) > EV(T_w;p)$ for $\alpha_{2k} < p < \alpha_{2k+1}$ and $EV(T_w;p) > EV(T_u;p)$ for $\alpha_{2k-1} < p < \alpha_{2k}$ (for appropriate k). It remains to show that T can be modified (if necessary) so that $EV(T_u;p) > EV(T_v;p)$ and $EV(T_w;p) > EV(T_v;p)$ for all $0 < p < 1$ and for all vertices $v \neq u, w$.

Suppose $EV(T_v;p) > EV(T_u;p)$ for some vertex $v \neq u, w$ and for some $0 < p < 1$. Now add $2M$ vertices (with sufficiently large M to be determined) of degree 1 to T so that M of the new vertices are adjacent only to u and the remaining M vertices are adjacent only to w. Call this new tree T', and note that $EV(T'_u;p) - EV(T'_w;p) = h(p)$ still holds.

Now $EV(T'_u;p) = EV(T_u;p) + Mp + Mp^2$, since the M new vertices adjacent to u contribute Mp to the polynomial and the other M vertices contribute Mp^2. Let r be the distance from v to the nearer of the two vertices u and w. This gives $EV(T'_v;p) = EV(T_v;p) + Mp^{r+1} + Mp^{r+2}$ for $r \geq 1$. Then

$$EV(T'_u;p) - EV(T'_v;p) = EV(T_u;p) - EV(T_v;p) + Mp + Mp^2 - Mp^{r+1} - Mp^{r+2}.$$

Now it is clear we can select M sufficiently large so that $EV(T'_u;p) - EV(T'_v;p) > 0$ for $0 < p < 1$.

Finally, note that $EV(T'_u;p) > EV(T'_x;p)$ for $0 < p < 1$, where x is one of the new vertices added to T (immediate if x is adjacent to u and only slightly harder if x is adjacent to w).

Repeat this procedure for all vertices $v \neq u, w$, comparing each vertex v to u and then to w. Let M be large enough so that $EV(T'_u;p) > EV(T'_v;p)$ and $EV(T'_w;p) > EV(T'_v;p)$ for all $0 < p < 1$ and all vertices $v \neq u, w$.

Thus, for any vertex v of T' ($v \neq w$), we have $EV(T'_u;p) > EV(T'_v;p)$ for $0 < p < 1$, and the same holds for $EV(T'_w;p) > EV(T'_v;p)$.

□

For example, if we let $h(p) = p(1-p)(1-4p)(1-2p)(3-4p)$, then we can construct a tree T with adjacent vertices u and w as in the proofs of Lemma 2.4 and Proposition 2.5. This gives $EV(T_u;p) = 5p + 2p^2 + 72p^3 + 2p^4 + 33p^5$ and

$EV(T_w;p) = 2p+27p^2+2p^3+82p^4+p^5$. The resulting tree has 115 vertices, (u,w) is an edge of T, and $u >_p w$ for $0 < p < \frac{1}{4}$ and $\frac{1}{2} < p < \frac{3}{4}$, while $w >_p u$ for $\frac{1}{4} < p < \frac{1}{2}$ and $\frac{3}{4} < p < 1$. See Figure 2 for a plot of the difference of these two polynomials.

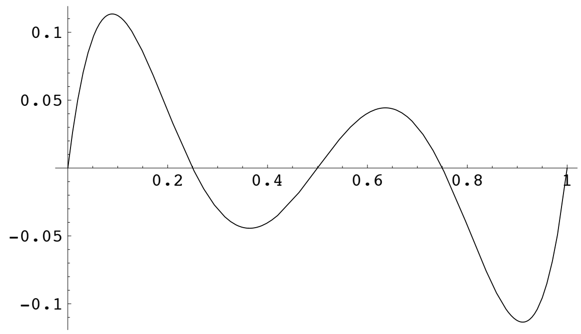

FIGURE 2. $EV(T_u;p) - EV(T_w;p)$.

We now generalize this result to show that ranking the top k choices for the root of a given graph can achieve an arbitrary permutation of k given vertices. We begin with a lemma that generalizes Lemma 2.4.

Lemma 2.6. *Let $f_k(p)$ be the probability that there is a path joining two distinct vertices in the complete graph K_k. Let $h_i(p) = (1 - f_k(p)) \sum_{j=1}^{r} c_{i,j}p^j$, where the $c_{i,j}$ are arbitrary integers and r is a positive integer. Then there is a graph G with a unique maximum clique $\{v_1, \ldots, v_k\}$ constructed by attaching trees T_i to each vertex v_i, $1 \le i \le k$, such that, for all $1 \le i \le k-1$,*

$$EV(G_{v_{i+1}};p) - EV(G_{v_i};p) = h_i(p).$$

Proof. Let G be a graph with clique $\{v_1, \ldots, v_k\}$ and with trees T_i so that vertex v_i is the only vertex from the clique that is in the tree T_i. See Figure 3 for an illustration. (To ease notation, we let T_i denote the rooted tree with root vertex v_i and let G_i denote the the graph G rooted at the vertex v_i.) Then

$$EV(G_i;p) = EV(T_i;p) + f_k(p)\sum_{j\neq i} EV(T_j;p) + (k-1)f_k(p),$$

so,

$$EV(G_{i+1};p) - EV(G_i;p) = (EV(T_{i+1};p) - EV(T_i;p))\,(1 - f_k(p)).$$

Our goal is to find positive integers $a_{i,j}$ so that, for each $1 \le i \le k-1$ and $1 \le j \le r$, we have $a_{(i+1),j} - a_{i,j} = c_{i,j}$. We fix j between 1 and r and proceed as follows. Initially set $a_{1,j} = 0$. This forces $a_{2,j} = c_{1,j}$, which forces $a_{3,j} = c_{1,j} + c_{2,j}$, which forces $a_{4,j} = c_{1,j} + c_{2,j} + c_{3,j}$, and so on, with $a_{l,j} = \sum_{i=1}^{l-1} c_{i,j}$ holding for all $l \le k$.

Some of these $a_{i,j}$ will be non-positive at this point (in particular, we have $a_{1,j} = 0$). To finish the assignment, let M be the minimum value among $\{a_{1,j}, \dots, a_{k,j}\}$, then add $|M| + 1$ to each $a_{i,j}$, forming a new sequence $a'_{i,j} = a_{i,j} + |M| + 1$. Each $a'_{i,j}$ will be a positive integer and will still satisfy $a'_{(i+1),j} - a'_{i,j} = c_{i,j}$ for all i and j. By Lemma 2.3, we can construct trees T_i having $EV(T_i; p) = \sum_{j=1}^{r} a'_{i,j} p^j$. This completes the proof.

□

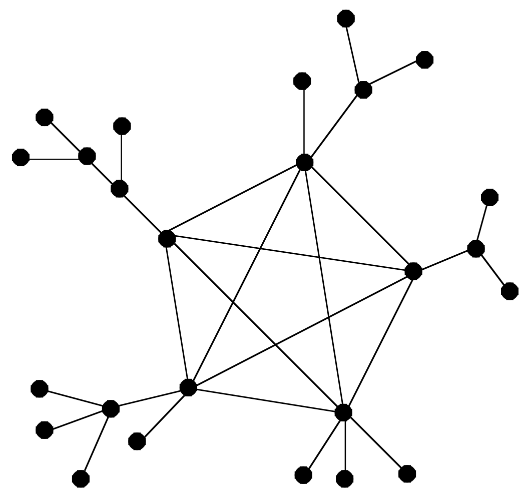

FIGURE 3. The construction used in Lemma 2.6 and Theorem 2.7.

We now give the main result in this section. Our construction uses Lemma 2.6 to generalize Proposition 2.5.

Theorem 2.7. *There is a graph G satisfying all of the following conditions:*

(1) $S = \{v_1, \dots, v_k\}$ *forms a maximum size clique in G;*
(2) *For any p between 0 and 1 and any vertices $v_i \in S$ and $w \notin S$, we have $v_i >_p w$;*
(3) *For each permutation $\pi \in S_k$, there is a value of p between 0 and 1 such that*

$$v_{\pi(1)} >_p v_{\pi(2)} >_p \cdots >_p v_{\pi(k)}.$$

Proof. We form G in stages. Initially, let $\{v_1, \dots, v_k\}$ be a clique. We will attach trees to each of the v_i as in Lemma 2.6. The proof will follow from the lemma when we show how to define polynomials $h_i(p)$ for $1 \le i \le k-1$ with specified roots.

More precisely, we will determine $k-1$ subsets R_i so that

- Each R_i is finite;
- Each $z \in R_i$ is a rational number between 0 and 1;
- $h_i(p) = Q(1-f_k(p))\prod_{z\in R_i}(z-p)$, where Q is an integer chosen to make all coefficients integers.

We now construct the R_i in stages.

(1) For all $1 \le i \le k-1$, $R_i \cap [0, \frac{1}{k!}] = \emptyset$. Thus, if $0 < p < \frac{1}{k!}$, we have $h_i(p) > 0$ for all i. This will force $v_k >_p v_{k-1} >_p \cdots >_p v_1$ for $0 < p < \frac{1}{k!}$.

(2) Let r_1 be a rational number with $\frac{1}{k!} < r_1 < \frac{2}{k!}$, and place $r_1 \in R_1$. Then $h_1(r_1) = 0$, and $h_1(p) < 0$ for $r_1 < p < \frac{2}{k!}$. This will force $v_k >_p v_{k-1} >_p \cdots >_p v_1 >_p v_2$ for $r_1 < p$ and $p - r_1$ sufficiently small.

(3) Now select r_2 rational with $\frac{2}{k!} < r_2 < \frac{3}{k!}$, and place $r_2 \in R_2$. As above, this will force $v_2 >_p v_3$ for $p \in (r_2, \frac{3}{k!})$. In addition, since $v_3 >_p v_1 >_p v_2$ when $p - r_1$ is small and $v_2 >_p v_3$ for $r_2 < p < \frac{3}{k!}$, we must have v_1 'pass' v_3 for some value of p between r_1 and r_2. Thus, we have the progression $v_3 >_p v_1 >_p v_2 \to v_1 >_p v_3 >_p v_2 \to v_1 >_p v_2 >_p v_3$ as p increases from $r_1 + \epsilon$ to $r_2 + \epsilon$.

(4) We continue in this fashion, adding r to R_i to change the sign of $h_i(p)$ as p increases from $r - \epsilon$ to $r + \epsilon$. This swaps the order of v_i and v_{i+1} in the induced order.

More generally, given a permutation π, we can record the orders of all pairs of consecutive indices. For example, if $\pi = 315624$, we require $v_1 >_p v_2, v_3 >_p v_2, v_3 >_p v_4, v_5 >_p v_4$ and $v_5 >_p v_6$. It is easy to ensure these pairwise relations hold by adding a rational z to the appropriate R_i. For instance, in our example, we wish to ensure $h_1(p), h_3(p)$ and $h_5(p) > 0$ and $h_2(p), h_4(p)$ and $h_6(p) < 0$ for a fixed p. Unfortunately, this does not uniquely determine π. There may be many total orders that extend the partial order determined by the pairwise relations.

In this case, however, we can add another rational to some R_i to effect a transposition. For instance, suppose we have 135624 instead of the desired $\pi = 315624$ for a value of p. Both of these orders are consistent with the pairwise relations. Then we add a new rational z to R_1. Then we must have v_1 pass v_3 for some $p < z$. By choosing an appropriate z, we can ensure the permutation 315624 will appear for this value of p.

Since every permutation $\pi \in S_k$ is a product of consecutive transpositions, we can achieve any permutation of the vertices $\{v_1, \dots, v_k\}$ by continuing the process. (While it is extremely tedious to compute each R_i, this process can be made rigorous with an inductive argument, generating all permutations of S_k by inserting k in all possible ways into the permutations of S_{k-1}.) Multiplying through and clearing denominators will produce polynomials with integer coefficients, as desired.

Finally, it remains to show that $v_i >_p x$ for any $x \notin S$ and for any value of p. As in the proof of Proposition 2.5, by attaching R vertices to each of the vertices $v_i \in S$, we do not affect the polynomial differences $h_i(p)$, so we do not change the relative ordering of the vertices. Then $v_i >_p x$ for any $x \notin S$ provided R is sufficiently large. We leave the details of determining an explicit value for R to the reader. □

We work through a slightly simplified version of the procedure described in the proof of Theorem 2.7 for $k = 3$. For this case, we can use $R_1 = \{\frac{1}{5}, \frac{3}{5}\}$ and $R_2 = \{\frac{2}{5}, \frac{4}{5}\}$. This gives $h_1(p) = 25(1 - f_3(p))\left(\frac{1}{5} - p\right)\left(\frac{3}{5} - p\right)$ and $h_2(p) = 25(1 - f_3(p))\left(\frac{2}{5} - p\right)\left(\frac{4}{5} - p\right)$. Then, using the algorithm given in the proof of Lemma 2.6, we attach trees to the vertices of K_3, where

$$\begin{aligned} EV(T_1;p) &= p + 51p^2 + p^3 \\ EV(T_2;p) &= 4p + 31p^2 + 26p^3 \\ EV(T_3;p) &= 12p + p^2 + 51p^3 \end{aligned}$$

The resulting graph will have 181 vertices, and every permutation of the three vertices of the clique will appear for some value of p. By construction, the order changes for $p = \frac{i}{5}$ for $i = 1, 2, 3$ or 4. Further, by the process described in Step (3) of the proof of Theorem 2.7, vertices 1 and 3 will swap places twice; once at $x = .326\ldots$ and again at $y = .673\ldots$ See Figure 4 for the correspondence between permutations and subintervals. In the figure, the sequence 321 represents the ordering $v_3 >_p v_2 >_p v_1$, and so on.

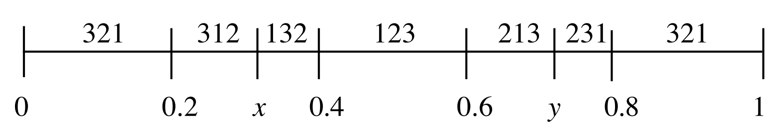

FIGURE 4. All permutations of three vertices appear as p increases from 0 to 1.

The order that we encounter the permutations in this procedure corresponds to a hamilton path in the poset on S_k under the *weak Bruhat order*. See Chapter 3 of [5] for more information.

3. Zeroes of $EV(G)$

We now turn to connections between algebraic properties of the polynomial and combinatorial properties of the network. Zeroes of the chromatic polynomial of a graph give information on the chromatic number, and complex zeroes give information about state changes in various percolation models in statistical physics [**?**, 16]. An interesting connection between graph coloring and analysis appears in [19]. Zeroes of the characteristic polynomial of a graph [7] (the eigenvalues of the graph) give information about a host of graph properties, especially connectivity. The next result relates the size of the largest rational zero of $EV(G_s; p)$ to the number of vertices of G.

Proposition 3.1. *Let r be the largest rational zero of $EV(G_s; p)$. Then G_s has at least r vertices (including the root).*

Proof. If $r < 2$, the result is immediate, so we assume $r \geq 2$. Recall that $EV(G_s; 1) = n - 1$, where n is the number of vertices of G. Now write $r = \frac{a}{b}$ (where $a \geq 2b$) and factor $EV(G_s; p) = (a - bp)h(p)$ for some polynomial $h(p)$ with integer coefficients. Then $h(1) = \frac{n-1}{a-b}$, so $(a - b) \leq n - 1$ since $h(1)$ must be an integer.

Case 1: r is an integer. Then $b = 1$, so $r = a$ and $a - 1 \leq n - 1$. Thus, G has at least r vertices.

Case 2: r is not an integer. It remains to show that $r \leq a - b$. Since r is not an integer, we have $b \geq 2$, so $\frac{b}{b-1} \leq 2$. Thus, $r = \frac{a}{b} > 2 \geq \frac{b}{b-1}$. Simplifying yields $a(b-1) - b^2 > 0$, or $a - b > \frac{a}{b} = r$, as desired.

In both cases, we have $r \leq n$, and this completes the proof.

□

The bound in Proposition 3.1 is sharp: Let G_s be the rooted graph consisting of one double edge and $n - 2$ single edges (where $n \geq 2$), all joined to the root. Then $EV(G_s; p) = p(n - p)$, and G_s has exactly n vertices. It is interesting to note that replacing the double edge with a 3-cycle has a dramatic effect on the location of the largest zero, which will be much smaller than the number of vertices.

When T is a tree, Proposition 3.1 tells us nothing, since, in that case, $EV(T_s; p)$ will be a polynomial with positive integer coefficients, thus having no positive zeroes. On the other hand, given any collection of negative rationals, it is trivial to construct a rooted tree T_s with $EV(T_s; p)$ having zeroes at precisely those given rationals. In particular, if $r_i = -\frac{a_i}{b_i}$, we can form the polynomial $f(p) = \prod_{i=1}^{d}(a_i + b_i p)$ to accomplish this.

Proposition 3.2. *Let* $\{r_1, \dots, r_d\}$ *be any collection of negative rational numbers. Then there is a rooted tree* T_s *such that* $EV(T_s; p)$ *has degree* d*, and has precisely the zeroes* $\{r_1, \dots, r_d\}$*.*

While rooted trees allow no positive zeroes, we can create rooted graphs with expected rank polynomials having virtually any collection of positive integer zeroes. We need the following definition:

Definition 3.3. The k^{th} *elementary symmetric function* in variables $x_1, \dots, x_d$ is

$$\sigma_k(x_1, \dots, x_d) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le d} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

By convention, $\sigma_0(x_1, \dots, x_d) = 1$.

Theorem 3.4. *Let* $\{r_1, \dots, r_d\}$ *be any collection of positive integers, let* $s_k = r_k - 1$ *for all* k*, and suppose* $\sigma_k(s_1, \dots, s_d) - \sigma_{k-1}(s_1, \dots, s_d) \ge 0$*, where* σ_k *is the* k^{th} *elementary symmetric function and* $1 \le k \le d$*. Then there is a rooted graph* G_s *with* $EV(G_s; p) = p \prod_{k=1}^{d} (r_k - p)$*.*

Proof. Let H^m denote the rooted graph with two vertices and m parallel edges, where $1 \le m \le d+1$. We construct a rooted graph G_s by gluing together a_m copies of the graph H^m for $1 \le m \le d+1$, where the non-negative integers a_m are determined below. To achieve this, we determine a_m for $1 \le m \le d+1$ such that

$$\sum_{m=1}^{d+1} a_m (1 - (1-p)^m) = p \prod_{k=1}^{d} (r_k - p).$$

Setting $y = 1-p$ and $s_k = r_k - 1$ and equating coefficients of the two polynomials, we get $a_{d+1} = 1$ and $a_m = \sigma_{d-m+1}(s_1, \dots, s_d) - \sigma_{d-m}(s_1, \dots, s_d)$ for $1 \le m \le d$. By supposition, $a_m \ge 0$ for $1 \le m \le d+1$, so the rooted graph G_s formed by identifying all of the roots of the a_m rooted graphs H^m for $1 \le m \le d+1$ will have $EV(G_s; p) = p \prod_{k=1}^{d} (r_k - p)$.

□

Example 3.5. We construct a rooted graph G_s with

$$EV(G_s; p) = p(5-p)(6-p)(7-p)(8-p)(9-p).$$

Expanding the polynomial as in the proof of Theorem 3.4, we find $a_1 = 776, a_2 = 3874, a_3 = 1715, a_4 = 325, a_5 = 29$ and $a_6 = 1$. Thus, if G_s is constructed by identifying the roots of a_m copies of the graph H^m, we have $EV(G_s; p) = p(5-p)(6-p)(7-p)(8-p)(9-p)$. This graph has 6721 vertices (including the root) and 15,120 edges.

While the previous construction is of limited practical value, we point out that the graph H^m can be thought of as a single, 'toughened' edge, where the probability of edge success is increased from p to $1-(1-p)^m$. Under this interpretation, the graph G_s constructed in Theorem 3.4 is simply a rooted star in which every vertex is adjacent to the root.

References

[1] M. Aivaliotis, G. Gordon, and W. Graveman, *When bad things happen to good trees*, J. Graph Theory **37** (2001), 79-99.

[2] A. Amin, K. Siegrist, and P. Slater, *Pair-connected reliability of a tree and its distance degree sequences*, Cong. Numer. **58** (1987), 29-42.

[3] A. Amin, K. Siegrist, and P. Slater, *Exact formulas for reliability measures for various classes of graphs*, Cong. Numer. **58** (1987), 43-52.

[4] A. Bailey, G. Gordon, M. Patton, and J. Scancella, *Expected value expansions in rooted graphs*, Discrete Appl. Math. **128** (2003), 555-571.

[5] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer, New York, 2005.

[6] J. Brown, C. Hickman, A. Sokal, and D. Wagner, *On the chromatic roots of generalized theta graphs*, J. Comb. Theory Ser. B **83** (2001), 272-297.

[7] F. R. K. Chung, *Spectral graph theory,* CBMS Regional Conference Series **92**, Amer. Math. Soc., Providence, 1997.

[8] C. Colbourn, *The combinatorics of network reliability*, Oxford University Press, Oxford, 1987.

[9] C. Colbourn, *Network resilience*, SIAM J. Alg. Disc. Meth. **8** (1987), 404-409.

[10] D. Eisenstat, G. Gordon, and A. Redlich, *Combinatorial properties of a rooted graph polynomial*, SIAM J. on Discrete Math. 22 (2008) 776-785.

[11] D. Eisenstat, J. Feder, G. Francos, G. Gordon, and A. Redlich, *Expected rank and randomness in rooted graphs*, Discrete Appl. Math. **156** (2008), 746-756.

[12] E. N. Gilbert, *Random graphs,* Ann. Math. Stat. **30** (1959), 1141-1144.

[13] A. K. Kelmans, *Connectivity of probabilistic networks*, Automation Remote Control **29** (1967), 444-460.

[14] A. K. Kelmans, *Crossing properties of graph reliability functions*, J. Graph Theory **35** (2000), 206-221.

[15] W. Myrvold, K. Cheung, L. Page, and J. Perry, *Uniformly-most reliable networks do not always exist*, Networks **21** (1991), 417-419.

[16] R. Shrock, *Chromatic polynomials and their zeroes and asymptotic limits for families of graphs*, Disc. Math. **231** (2001), 421-446.

[17] K. Siegrist, *Expected value expansions in random subgraphs with applications to network reliability*, Comb., Probab. Comput. **7** (1993), 465-483.

[18] K. Siegrist, A. Amin, and P. Slater, *The optimal unicyclic graphs for pair-connected reliability*, Disc. Appl. Math. **41** (1993), 235-243.

[19] W. T. Tutte, *On chromatic polynomials and the golden ratio*, J. Comb. Theory **9** (1970), 289-296.

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