MATROID AUTOMORPHISMS OF THE ROOT SYSTEM H_4

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ABSTRACT. We study the rank-4 linear matroid $M(H_4)$ associated with the 4-dimensional root system H_4. This root system coincides with the vertices of the 600-cell, a 4-dimensional regular solid. We determine the automorphism group of this matroid, showing half of the 14,400 automorphisms are geometric and half are not. We prove this group is transitive on the flats of the matroid, and also prove this group action is primitive. We use the incidence properties of the flats and the *orthoframes* of the matroid as a tool to understand these automorphisms, and interpret the flats geometrically.

1. INTRODUCTION

Regular polytopes in 4-dimensions are notoriously difficult to understand geometrically. Coxeter's classic text [3] is an excellent resource, concentrating on both the metric properties and the symmetry groups of regular polytopes. Another approach to understanding these polytopes is through combinatorics; we use matroids to model the linear dependence of a collection of vectors associated to the polytope. That is the context for this paper, and we concentrate on the matroid associated with the 120-cell or the 600-cell, two dual 4-dimensional regular polytopes.

The connection between polytopes and matroids, or, more generally, between root systems and matroids, is as follows. Given a finite set S of vectors in $\mathbb{R}^n$ possessing a high degree of symmetry, define the (linear) matroid $M(S)$ as the dependence matroid for the set S over $\mathbb{R}$. Then there should be a close relationship between the symmetry group of the original set S (*geometric* symmetry) and the matroid automorphism group $\text{Aut}(M(S))$ (*combinatorial* symmetry). In particular, every geometric symmetry necessarily preserves the dependence structure of S, so gives rise to a matroid automorphism.

The root system H_4 can be obtained by choosing the 120 vectors in $\mathbb{R}^4$ that form the vertices of the 600-cell. These vectors come in 60 pairs, and

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each pair corresponds to a single point in the matroid. Thus, $M(H_4)$ is a rank-4 matroid on 60 points.

This paper generalizes and extends [5]. In particular, we are interested in understanding the structure of the matroid automorphism group $\text{Aut}(M(H_4))$. We show (Theorem 4.6) that $\text{Aut}(M(H_4))$ contains *non-geometric* automorphisms in the sense that half of the 14,400 elements of $\text{Aut}(M(H_4))$ do not arise from the Coxeter/Weyl group $W(H_4)$. We also prove the automorphism group of $M(H_4)$ acts transitively on each class of flats of the matroid (Lemma 4.2), and that the action is primitive (Theorem 4.4). A key tool for understanding the structure of the automorphisms is the incidence relation among the flats of $M(H_4)$ (Lemma 3.2 and Proposition 3.3). This incidence structure allows us to compute the stabilizer of a point of the matroid (Lemma 4.1), a fact we need to understand the structure of the group.

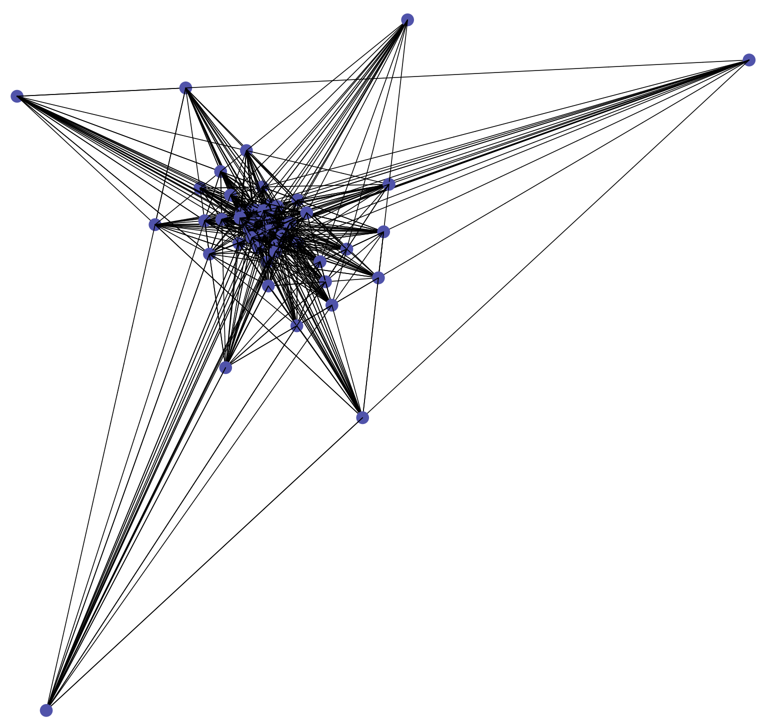

FIGURE 1. A projection of the matroid $M(H_4)$.

The connection between the geometric and combinatorial symmetry of certain root systems has been explored in [4, 5, 6, 7]. In [7], matroid automorphism groups are computed for the root systems A_n, B_n and D_n, while [5] considers the root system H_3 associated with the icosahedron and [6] examines the matroid associated with the root system F_4. The general case is treated in [4], where a computer program is employed to show that $\text{Aut}(M(S)) \cong G_S/W$ for all root systems S *except* F_4, H_3 and H_4, where G_S is the Coxeter/Weyl group associated with S and W is either the 2-element group $\mathbb{Z}_2$ (when G has *central inversion*) or W is trivial (when G does not have central inversion). No attempt is made to understand the structure of these matroids in [4], however.

Other models for connecting geometric and combinatorial symmetry are possible, of course. In particular, since each pair of vectors $\pm\mathbf{v}$ in a root system corresponds to a double point in the associated linear matroid, we could consider both vectors in the matroid. This has the effect of doubling the number of automorphisms for each such pair; in our case, this increases the number of automorphisms by a factor of 2^{60}. Alternatively, we could, associate an *oriented* matroid with the root system. This doubles the number of automorphisms considered here. Another option is to consider a projective version of the root system. We point out, however, that all of these modifications differ from our treatment in transparent ways that do not change our understanding of the connection between the geometry and the combinatorics.

This paper is organized as follows: The matroid $M(H_4)$ is defined as the column dependence matroid for a 4×60 matrix in Section 2. In Section 3, we describe the flats and *orthoframes* of the matroid and their incidence. Orthoframes are special bases of the matroid, and they are important for understanding a certain kind of duality between points and 15-point planes. This point-plane correspondence is made explicit in Propositions 3.5(4), 3.8, and 5.1, where it is interpreted combinatorially, algebraically and geometrically, respectively. Orthoframes also allow us to reconstruct the matroid - Proposition 3.7.

Section 4 is the heart of this paper, concentrating on the structure of the matroid automorphisms. We show that the stabilizer of a point x is $\text{stab}(x) \cong S_5 \times \mathbb{Z}_2$ (Lemma 4.1), then use this to show that $\text{Aut}(M(H_4))$ acts transitively on flats (Lemma 4.2) and primitively on the matroid (Theorem 4.4). This allows us to understand the structure of the group - Theorem 4.6. We conclude (Section 5) with a few connections between the flats of the matroid and various classes of faces of the 120- and 600-cell.

We would like to thank Derek Smith and David Richter for useful discussions about the Coxeter/Weyl group W for the H_4 root system. The third author especially thanks Prof. Thomas Brylawski for teaching him about matroids and the beauty of symmetry groups.

2. Preliminaries

We assume some basic familiarity with matroids and root systems. We refer the reader to the first chapter of [10] for an introduction to matroids and [8, 9] for much more on root systems. The study of root systems is very important for Lie algebras, and the term 'root' can be traced to characteristic roots of certain Lie operators. For our purposes, the collection of roots forms a matroid, and the Coxeter/Weyl group of the root system is closely related to the automorphism group of that matroid.

The root system H_4 has an interpretation via two dual 4-dimensional regular polytopes, the 120-cell and the 600-cell. The 120-cell is composed of 120 dodecahedra and the 600-cell is composed of 600 tetrahedra. Each vertex of the 120-cell is incident to precisely 3 dodecahedra and each vertex of the 600-cell meets 5 tetrahedra, justifying the intuitive notion that the 120-cell is a 4-dimensional analogue of the dodecahedron while the 600-cell is a 4-dimensional version of the icosahedron.

As dual polytopes, the 120-cell and the 600-cell have the same set of hyperplane reflections and symmetry groups. Then the connection between these two dual solids and the root system H_4 is direct: The roots are precisely the normal vectors of all the reflecting hyperplanes that preserve the 120-cell (or, dually, the 600-cell). A copy of this root system also appears as the collection of 120 vertices of the 600-cell (where the 600-cell is positioned with the origin at its center and we identify a vertex with the vector from the origin to that vertex). Extensive information about these polytopes appears in Table 5 of the appendix of [3].

Definition 2.1. The matroid $M(H_4)$ is defined to be the linear dependence matroid on the set of 60 column vectors of the matrix H over $\mathbb{Q}[\tau]$, where $\tau = \frac{1+\sqrt{5}}{2}$ satisfies $\tau^2 = \tau + 1$.

$$H = \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{array}\right. \cdots$$

$$\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tau & \tau & \tau & \tau & \tau^2 & \tau^2 & \tau^2 & \tau^2 & 1 & 1 & 1 & 1 \\ \tau^2 & \tau^2 & -\tau^2 & -\tau^2 & 1 & 1 & -1 & -1 & \tau & \tau & -\tau & -\tau \\ 1 & -1 & 1 & -1 & \tau & -\tau & \tau & -\tau & \tau^2 & -\tau^2 & \tau^2 & -\tau^2 \end{array} \cdots$$

$$\begin{array}{cccccccccccc} \tau & \tau & \tau & \tau & \tau^2 & \tau^2 & \tau^2 & \tau^2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & \tau & \tau & -\tau & -\tau & \tau^2 & \tau^2 & -\tau^2 & -\tau^2 \\ \tau^2 & -\tau^2 & \tau^2 & -\tau^2 & 1 & -1 & 1 & -1 & \tau & -\tau & \tau & -\tau \end{array} \cdots$$

$$\begin{array}{cccccccccccc}
\tau & \tau & \tau & \tau & \tau^2 & \tau^2 & \tau^2 & \tau^2 & 1 & 1 & 1 & 1 \\
\tau^2 & \tau^2 & -\tau^2 & -\tau^2 & 1 & 1 & -1 & -1 & \tau & \tau & -\tau & -\tau \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & \tau & -\tau & \tau & -\tau & \tau^2 & -\tau^2 & \tau^2 & -\tau^2
\end{array} \cdots$$

$$\left.\begin{array}{cccccccccccc}
\tau & \tau & \tau & \tau & \tau^2 & \tau^2 & \tau^2 & \tau^2 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & \tau & \tau & -\tau & -\tau & \tau^2 & \tau^2 & -\tau^2 & -\tau^2 \\
\tau^2 & -\tau^2 & \tau^2 & -\tau^2 & 1 & -1 & 1 & -1 & \tau & -\tau & \tau & -\tau \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

The full root system H_4 consists of these 60 column vectors together with their 60 negatives. Note that replacing any column vector by its negative does not change the matroid. See Sec. 8.7 of [3] for more on the derivation of these coordinates.

Since $r(M(H_4)) = 4$, we can represent the matroid with an affine picture in $\mathbb{R}^3$. We find affine coordinates in $\mathbb{R}^3$ as follows: First, find a non-singular linear transformation of the column vectors of H_4 that maps each vector to an ordered 4-tuple in which the first entry is non-zero, then project onto the plane $x_1 = 1$ and plot the remaining ordered triples. We note that choosing different transformations gives rise to different projections; choosing the 'best' such projection is subjective. In Figure 1, we give a projection of one such representation.

3. The flats and orthoframes of $M(H_4)$

We describe the rank-4 matroid $M(H_4)$ by determining the number of flats of each kind and the flat incidence structure. This incidence structure will also be important for determining the automorphisms of the matroid. We use lower case letters to label the points of the matroid and upper case letters for flats of rank 2 or 3.

3.1. **Flats.** Every line in $M(H_4)$ has 2, 3 or 5 points, and there are 4 different isomorphism classes of planes (rank-3 flats) in $M(H_4)$. The planes are shown in Figure 2. This fact can be proven by a direct computation using the column dependences of the matrix H.

Lemma 3.1. *Flat counts: In Table 1, we list the number of flats of rank 1, 2 and 3 in the matroid $M(H_4)$.*

Diagrams of $M(H_4)$ that emphasize the 3-point lines and 5-point lines appear in Figure 3.

Lemma 3.2. *Flat incidence: In Table 2, we list the number of flats of a certain kind that contain a given flat of lower rank.*

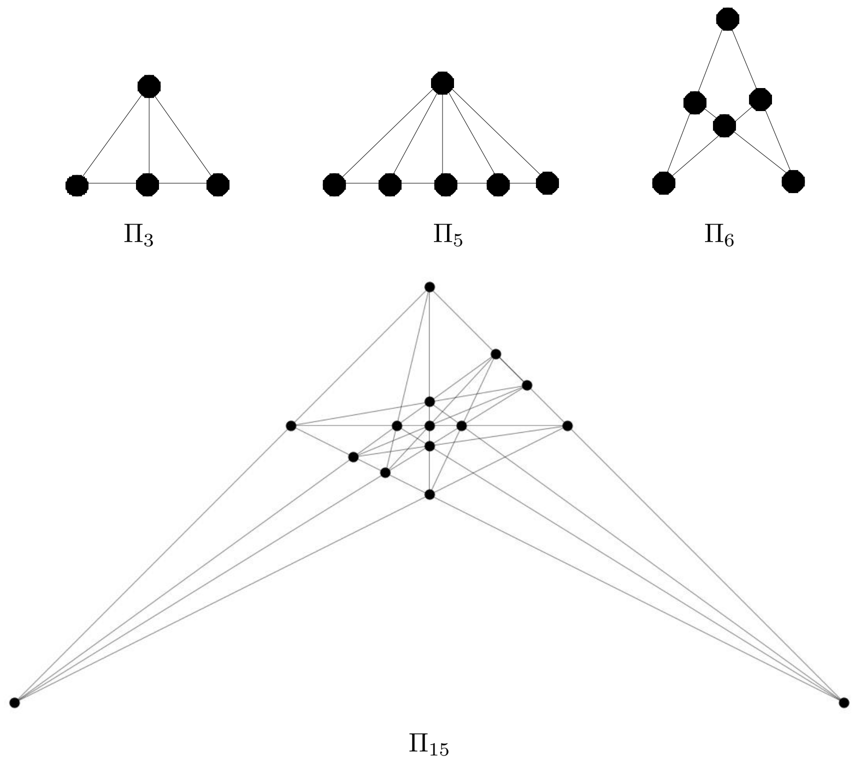

Figure 2. The four planes that appear in $M(H_4)$.

Table 1. The number of flats of each kind in the matroid.

Rank	Rank 1	Rank 2			Rank 3			
Flat	Points	2-pt lines	3-pt lines	5-pt lines	Π_3	Π_5	Π_6	Π_{15}
No.	60	450	200	72	600	360	300	60

Both lemmas can be verified by computer calculations, but we give an example of how the various counts are interrelated. Assuming the point-flat incidence counts for 3-point lines and Π_{15} planes, we will count the number of Π_6 planes; other counts may be obtained with similar arguments.

For a given point $x \in M(H_4)$, there are ten 3-point lines through x, giving 45 pairs of 3-point lines containing x. Now x is in 15 Π_{15} planes, and each of these planes is completely determined by the pair of 3-point lines containing x. Each of the remaining 30 pairs of 3-point lines containing

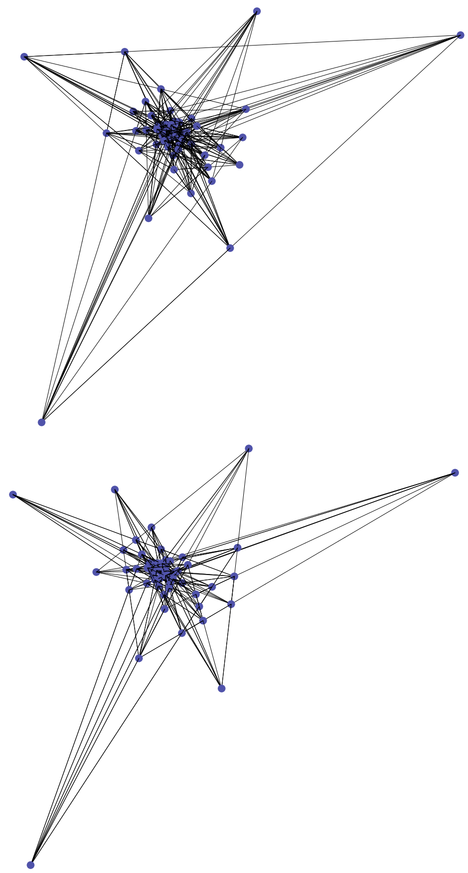

FIGURE 3. The 3-point lines (left) and 5-point lines (right) of $M(H_4)$.

	Rank 2			Rank 3			
	2-pt lines	3-pt lines	5-pt lines	Π_3	Π_5	Π_6	Π_{15}
Point	15	10	6	$10^{(a)}$	$6^{(a)}$	30	15
2-pt line	1	-	-	4	4	2	2
3-pt line	-	1	-	3	-	6	3
5-pt line	-	-	1	-	5	-	5

TABLE 2. The number of flats of one kind that contain a given flat of another kind. (a) The point is the apex of the Π_3 or Π_5.

x uniquely determine a Π_6 containing x. Thus, there are 30 Π_6 planes containing a given point.

To get the total number of Π_6 planes, consider the point-Π_6 incidence. Each point is in 30 Π_6 planes, and each Π_6 contains 6 points. Thus, the total number of Π_6 planes is 300.

The flats of a matroid satisfy the *flat covering property*:

> If F is a flat in a matroid M, then $\{F'-F \mid F' \text{ is a flat that covers } F\}$ partitions $E-F$.

We illustrate this partitioning property for $M(H_4)$:

Point/line incidence: From Table 2, we know a given point x is covered by precisely 15 2-point lines, ten 3-point lines and six 5-point lines. Then it is easy to see this pencil of lines contains precisely 59 points (not counting x), partitioning $E-x$, as required.

Line/plane incidence: We consider the three kinds of lines in $M(H_4)$.

- 2-point lines: Each 2-point line L is covered by four Π_3's, four Π_5's, two Π_6's and two Π_{15}'s. Each Π_3 that covers L contains two points not on L. Similarly, each Π_5 covering L has four more points, each such Π_6 also has four more points, and each such Π_{15} has 13 points. This gives us the required partition of the remaining 58 points.
- 3-point lines: Each 3-point line L is in 3 Π_3's, 6 Π_6's and 3 Π_{15}'s. As above, counting the points in these covering planes gives a total of 57 points partitioned by these planes.
- 5-point lines: If L is a 5-point line, then only two kinds of planes contain L: the 5 Π_5's and the 5 Π_{15}'s. These 10 planes contain 55 points (excluding the points on L), again giving us the required partition of $E-L$.

As an application of the incidence data given above, we prove the following.

Proposition 3.3. *Every pair of* Π_{15} *planes intersect.*

Proof. Let P be a 15-point plane and let L_5 be a 5-point line contained in P. Since every 5-point line is contained in precisely five Π_{15}'s, there are four Π_{15}'s that meet P along the line L_5. Since P contains six 5-point lines, this gives a total of 24 Π_{15}'s that meet our given plane P in a 5-point line.

We repeat this argument for 3-point lines: Each of the ten 3-point lines in P is contained in two more Π_{15}'s, accounting for another 20 Π_{15}'s meeting P.

Finally, each two-point line is in two Π_{15}'s, but there are 15 2-point lines in P. This gives another 15 Π_{15}'s planes that meet P in a 2-point line. But this now accounts for 59 Π_{15}'s, all of which meet P in either a 2, 3 or 5-point line. Thus, every pair of Π_{15}'s meet.

□

In fact, all of these intersections are *modular*: $r(P_1 \cap P_2) = 2$ for all pairs of 15-point planes P_1 and P_2. We also remark the 15-point planes are isomorphic (as matroids) to $M(H_3)$ - the matroid associated to the root system H_3 (see [5]). We will need this connection in Section 4.

3.2. **Orthoframes.** Of special interest is the interesting symmetry between points and Π_{15} planes: There are 60 points and 60 Π_{15}'s, where each point is in 15 Π_{15}'s and each Π_{15} has 15 points. The easiest way to understand this symmetry is through *orthoframes*.

Definition 3.4. A basis B for $M(H_4)$ is an *orthoframe* if each pair of points in B forms a 2-point line in the matroid.

For instance, the basis formed by the first 4 columns of the matrix H is an orthoframe. In general, these bases correspond to column vectors in H that are pairwise orthogonal. Two more orthoframes are:

$$\begin{bmatrix} 0 & 1 & \tau & \tau^2 \\ 1 & 0 & \tau^2 & -\tau \\ -\tau & \tau^2 & 0 & -1 \\ -\tau^2 & -\tau & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & \tau^2 \\ 1 & \tau & \tau & -\tau \\ 1 & -\tau^2 & 0 & -1 \\ 1 & 1 & -\tau^2 & 0 \end{bmatrix}$$

Orthoframes are important to us for two reasons: matroid automorphisms give group actions on the set of orthoframes, and orthoframes have an immediate geometric interpretation in the root system H_4.

We state (without proof) several useful facts we will need about orthoframes. The proofs are routine, and follow in a similar way the incidence counts of Lemmas 3.1 and 3.2.

Proposition 3.5.
(1) *B is an orthoframe if and only if the four column vectors corresponding to the points of B are pariwise orthogonal.*
(2) *There are 75 orthoframes.*
(3) *Each point is in 5 orthoframes, and each 2-point line is in exactly one orthoframe.*
(4) *If $O_1, O_2, \ldots, O_5$ are the 5 orthoframes that contain a given point x, then $\bigcup_{i=1}^{5} O_i - x$ is a 15-point plane.*

Part (4) of this proposition allows us to define a bijection between the points of the matroid and the 15-point planes: Given a point x, let $O_1, O_2, \ldots, O_5$ be the five orthoframes that contain x. Then define $P_x := \bigcup_{i=1}^{5} O_i - x$. Conversely, a given 15-point plane can be partitioned into five partial orthoframes (this partition is visible in the picture of a Π_{15} in Fig. 2 – see also Sec. 2.1 of [5]). Then a 15-point plane P_x uniquely determines a point x that "completes" each of these orthoframes.

We will use this correspondence frequently; we introduce some terminology suggestive of the relationship between the column vectors corresponding to the point and the plane.

Definition 3.6. Suppose the point x corresponds to the 15-point plane P_x as above. Then we say the point x is the *orthopoint* of the plane P_x and the 15-point plane P_x is the *orthoplane* of the point x.

We can also use the orthoframes to uniquely reconstruct the matroid $M(H_4)$.

Proposition 3.7. *The collection of 75 orthoframes completely determines all the flats of the matroid* $M(H_4)$.

Proof. We show how the orthoframe data allows us to reconstruct all the flats.

- Π_{15} **planes:** The union of the orthoframes containing a given point x form the 15-point orthoplane P_x (where the common point is removed), so we can construct all the Π_{15}'s this way.
- **Lines:** Since each 2-point line is in a unique orthoframe, we simply list the six 2-point lines contained in each of the 75 orthoframes, giving us the 450 2-point lines. By the proof of Prop. 3.3, every 3-point line and every 5-point line occurs as the intersection of some pair of Π_{15}'s. This allows us to reconstruct all rank-2 flats.
- Π_3 **and** Π_5 **planes:** For the trivial planes Π_3 and Π_5, each such plane arises as the union of a 3 or 5-point line in a Π_{15} with the plane's orthopoint as the apex of the Π_3 or Π_5.
- Π_6 **planes:** The remaining non-trivial flats are the 300 Π_6 planes. We consider all pairs of intersecting 3-point lines. Each intersecting pair determines either a Π_{15} or a Π_6. We know all the 15-point planes at this point, so we can determine all pairs giving a Π_6. To reconstruct each Π_6 from this information, note that each Π_6 contains four 3-point lines, every pair of which intersect. This allows us to uniquely determine each Π_6 from the collection of 3-point lines.

□

Compared with bases, orthoframes provide a much more efficient way to describe the matroid. While there are 75 orthoframes, a computer search gives 398,475 bases; a random subset of four columns has approx 81.7% chance of being a basis.

We conclude this section by noting an algebraic explanation for the point-orthoplane correspondence. Each point corresponds to an ordered 4-tuple

$[a, b, c, d]$, and each Π_{15} corresponds to the solution set of a linear equation. The connection between the coordinates of the point z and the corresponding linear equation the associated orthoplane P_z satisfies is simple.

Proposition 3.8. *Let z be a point with corresponding orthoplane P_z, and suppose z corresponds to the ordered 4-tuple $[a, b, c, d]$. Then P_z is defined by the linear equation $ax_1 + bx_2 + cx_3 + dx_4 = 0$.*

Proof. Let z be a point and let $O_1, O_2, \dots, O_5$ be the five orthoframes containing z. Then if $y \in \bigcup_{i=1}^{5} O_i - z = P_z$, we have the 4-tuples corresponding to the points y and z are orthogonal (by Prop. 3.5(1)). Thus, if the coordinates for z are $[a, b, c, d]$, we have $ax_1 + bx_2 + cx_3 + dx_4 = 0$ for all column vectors $[x_1, x_2, x_3, x_4] \in P_z$. □

As an example of this algebraic connection, let z be the point with coordinates $[\tau^2, 0, \tau, -1]$. Then the equation $\tau^2 x_1 + \tau x_3 - x_4 = 0$ is satisfied by P_z, where P_z consists of the following 15 points:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & \tau & 1 & \dots \\ 1 & 1 & -1 & \tau^2 & \tau^2 & 1 & 1 & 0 & 0 & \dots \\ 0 & -1 & -1 & 1 & -1 & \tau & -\tau & -1 & -\tau^2 & \dots \\ 0 & 1 & 1 & \tau & -\tau & \tau^2 & -\tau^2 & \tau^2 & -\tau & \dots \end{pmatrix.$$
$$\begin{matrix} \dots & 1 & 1 & \tau & \tau & 1 & 1 \\ \dots & \tau & -\tau & 1 & -1 & \tau^2 & -\tau^2 \\ \dots & 0 & 0 & -\tau^2 & -\tau^2 & -\tau & -\tau \\ \dots & \tau^2 & \tau^2 & 0 & 0 & 0 & 0 \end{matrix}\Bigg).$$

4. AUTOMORPHISMS

We turn to our main topic: the structure of the automorphisms of $M(H_4)$. For a group G acting on a set X with $x \in X$, recall the *stabilizer of* x

$$\text{stab}(x) = \{g \in G \mid g(x) = x\}.$$

Lemma 4.1. *Let x be a point of $M(H_4)$ and P_x its 15-point orthoplane. Then* $\text{stab}(x) = \text{stab}(P_x) \cong S_5 \times \mathbb{Z}_2$.

Proof. The point-orthoplane correspondence (Prop. 3.5(4) or Prop. 3.8) gives $\text{stab}(x) = \text{stab}(P_x)$. Note that $P_x \cong M(H_3)$, the matroid associated with the icosahedral root system. Then, by Theorem 3.3 of [5], $\text{Aut}(M(H_3)) \cong S_5$, so S_5 fixes the plane P_x. (S_5 acts on the five rank-3 orthoframes.) Thus $S_5 \leq \text{stab}(x)$.

We may now suppose $\sigma \in \text{stab}(P_x)$ where σ fixes the orthoplane P_x pointwise. We will show that $\sigma = I$ or σ is the matroid automorphism

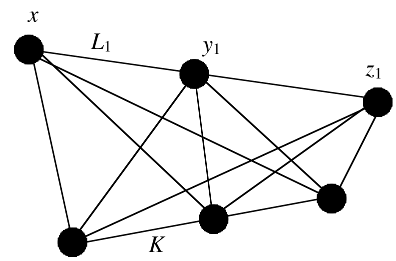

FIGURE 4. The three points x, y_1 and z_1 forming the apexes of the three Π_3's containing the line K are collinear. K and L_1 are skew, i.e, $r(K \cup L_1) = 4$.

induced by geometric reflection r_x of the root system through P_x. (Note that reflection fixes P_x pointwise, but also fixes the orthopoint x.)

So assume $\sigma(w) = w$ for $w = x$ and for all $w \in P_x$. Then σ fixes (at least) 16 points; we partition the remaining 44 points of the matroid into two classes:

Class 1: Let $\{L_1, L_2, \ldots, L_{10}\}$ be the pencil of 3-point lines through x. Then $\mathcal{C}_1 := \bigcup L_i - x$ contains 20 points. We write $L_i = \{x, y_i, z_i\}$ for $1 \leq i \leq 10$, so $\mathcal{C}_1 = \{y_1, y_2, \ldots, y_{10}, z_1, z_2, \ldots, z_{10}\}$. For a given i, we first show σ either fixes both y_i and z_i or it swaps them.

Consider a 3-point line K in the 15-point plane P_x, which we know is fixed pointwise by σ. Then K is also fixed pointwise. The line K is contained in three Π_3's (Lemma 3.2), with three different apexes, one of which is x. Then it is straightforward to show these three apexes form a 3-point line, so they correspond to one of the lines, say L_1, in the pencil through x, as in Figure 4. Since matroid automorphisms preserve all Π_3's, and since K is fixed, we must have $\sigma(y_1) \in \{y_1, z_1\}$.

But there are ten 3-point lines in P_x, and each of these lines will correspond to one of the L_j in precisely the same way K corresponds to L_1. Thus, we have $\sigma(y_i) \in \{y_i, z_i\}$ for all i.

Now suppose $\sigma(y_1) = y_1$. We will show that $\sigma(y_i) = y_i$ for all i (and so σ is the identity on $\mathcal{C}_1$). Now every pair of lines L_i, L_j determines either a 6-point plane Π_6 or a 15-point plane Π_{15}. Our incidence counts from Lemma 3.2 can be used to show that, for a given i, precisely 6 lines L_j can be paired with L_i to generate a Π_6, and the remaining three lines will generate Π_{15}'s when paired with L_i. We concentrate on the Π_6's.

Suppose L_1 and L_2 determine a Π_6, where w is the unique point of the Π_6 not on L_1 or L_2, as in Figure 5. Since $\sigma(x) = x$ and

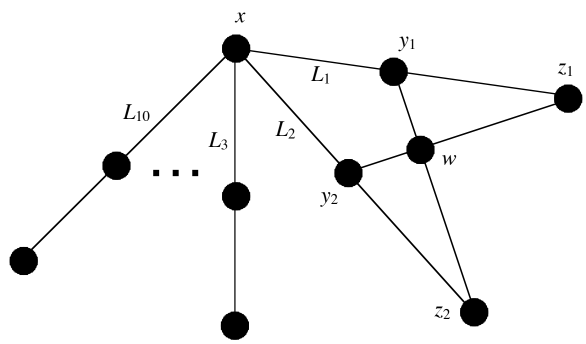

Figure 5. The pencil of 3-point lines through x. L_1 and L_2 generate a Π_6.

$\sigma(y_1) = y_1$, we know $\sigma(z_1) = z_1$. Thus, if σ swaps y_2 and z_2, then the 3-point line $\{y_1, w, z_2\}$ is mapped to the independent set $\{y_1, w, y_2\}$, which is impossible for a matroid automorphism. Thus, σ fixes y_2 and z_2.

To show that σ fixes all y_i and z_i, construct a graph Γ as follows: The 10 vertices are labeled by the lines L_i, with an edge between L_i and L_j if and only if these two lines determine a Π_6. Then Γ is a regular graph on 10 vertices with every vertex having degree 6, so Γ is connected. Thus we can find a path from L_1 to any line L_j, and it is clear that each edge of the path forces σ to fix the points on the corresponding line. Thus, σ fixes each point in $\mathcal{C}_1$. (Incidentally, we note the point w is on the fixed 15-point plane P_x. By choosing different pairs of lines in the pencil, we can locate all 15 points of P_x in this way.)

Finally, if σ swaps any pair y_i, z_i, then σ swaps *all* pairs, by a similar argument. Then σ corresponds to the reflection r_x through P_x.

Class 2: Let $\{M_1, M_2, \ldots, M_6\}$ be the pencil of six 5-point lines through x (again, from Lemma 3.2). Then $\mathcal{C}_2 := \bigcup M_i - x$ contains 24 points. As we did for $\mathcal{C}_1$, we show that these 24 points are either swapped in 12 transpositions (when σ corresponds to reflection) or are all fixed pointwise (when $\sigma = I$).

As before, fix a 5-point line K in the fixed plane P_x and consider the five points in the matroid that form the apexes of Π_5 planes which use K. Then it is again straightforward to show that these five apexes form a 5-point line, so they correspond to one of the M_i. (This is completely analogous to the situation with Π_3's that contain a fixed 3-point line, as in Figure 4.) Since matroid automorphisms preserve Π_5's, each line M_i in the pencil must be fixed.

We need to show that $\mathcal{C}_2$ is fixed by σ when σ fixes $\mathcal{C}_1$ pointwise. Now the 15 pairs of lines in the pencil $\{M_1, M_2, \ldots, M_6\}$ generate the 15 Π_{15} planes containing x. Thus, if σ fixes $\mathcal{C}_1$ pointwise, it fixes two intersecting 3-point lines in each of these Π_{15}'s, since the Π_{15}'s containing x are also generated by 15 pairs of lines from $\{L_1, L_2, \ldots, L_{10}\}$. Thus, in each Π_{15} that contains x, we have a pair of intersecting 3-point lines that are fixed pointwise, and a pair of intersecting 5-point lines that are also fixed (not necessarily pointwise).

But the only automorphism of a 15-point plane with this cycle structure on its 3- and 5-point lines is the identity – this follows from the last two columns of Table 1 of [5]. Thus, σ fixes $\mathcal{C}_2$ pointwise.

If σ swaps each pair (y_i, z_i) in $\mathcal{C}_1$, then we obtain reflection again, and the 24 points in $\mathcal{C}_2$ are all moved in 12 transpositions, corresponding to the reflection r_x through the plane P_x.

Thus, every $\sigma \in \text{stab}(x)$ can be decomposed as an automorphism of the plane P_x followed or not by reflection through that plane. These two operations commute, so we have $\text{stab}(x) \cong S_5 \times \mathbb{Z}_2$.

□

Recall there are seven different equivalence classes of flats: 2, 3 and 5-point lines, and 4 different classes of planes.

Lemma 4.2. $\text{Aut}(M(H_4))$ *acts transitively on each equivalence class of flats of the matroid.*

Proof. The proof makes use of the fact that the Coxeter/Weyl group acts transitively on the roots of H_4 (see [3]). Since every geometric symmetry of the root system gives rise to a matroid automorphism, we immediately get $\text{Aut}(M(H_4))$ acts transitively on the points of the matroid. The point-orthoplane correspondence then gives us a transitive action on the Π_{15}'s.

We now consider the remaining flat classes.

Rank 2 flats: Let $\mathcal{L}_k$ be the class of all k-point lines, for $k = 2, 3$ and 5, and let L_1 and L_2 be two k-point lines. If L_1 and L_2 are both in the same Π_{15}, then we use the fact (see [5]) that $\text{Aut}(M(H_3))$ acts transitively on lines to get an automorphism σ mapping L_1 to L_2.

If L_1 and L_2 are not contained in any Π_{15}, then either $r(L_1 \cup L_2) = 4$, i.e., the lines L_1 and L_2 are skew, or L_1 and L_2 are 3-point lines in a Π_6. In the former case, find two 15-point planes P_1 and P_2 with $L_1 \subseteq P_1$ and $L_2 \subseteq P_2$. Now use the transitivity on 15-point

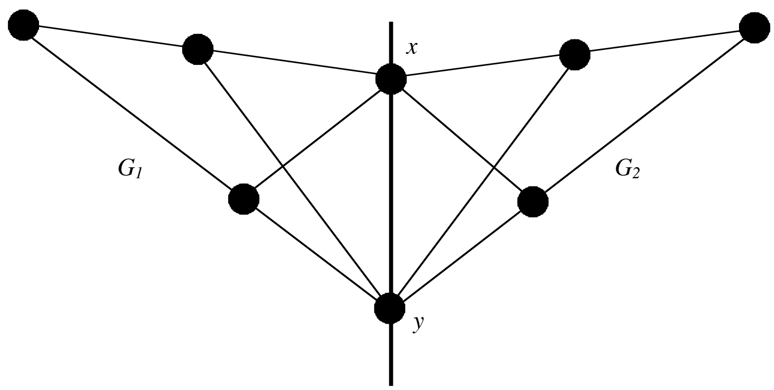

FIGURE 6. Two Π_6 planes share the 2-point line xy. $r(G_1 \cup G_2) = 4$.

planes to map P_1 to P_2, and then use transitivity on lines within P_2 to map the image of L_1 to L_2.

If $L_1 = \{a, b, c\}$ and $L_2 = \{a, d, e\}$ are intersecting 3-point lines in a Π_6, then use transitivity on points to map b to d. This must carry L_1 to L_2.

Rank 3 flats: We already have $\mathrm{Aut}(M(H_4))$ is transitive on 15-point planes. It is also clear that transitivity of 3- and 5-point lines gives us transitivity on Π_3 and Π_5 planes. It remains to prove transitivity for Π_6 planes.

Let G_1 and G_2 be two Π_6 planes, and let L_1 and L_2 be 2-point lines with $L_i \subseteq G_i$ ($i = 1$ or 2). Then transitivity on 2-point lines allows us to map $L_1 \mapsto L_2$. So we can assume G_1 and G_2 share the 2-point line xy, as in Figure 6. By Lemma 3.2, G_1 and G_2 are the only two Π_6's that contain xy. Then there are two 15-point planes that also contain the 2-point line xy; call these two planes P_1 and P_2. Then reflecting through either P_1 or P_2 will map G_1 to G_2, since reflection must send a Π_6 to a Π_6, reflections move 44 points, and G_1 and G_2 are the only two Π_6's containing xy.

□

As an example of how transitivity on Π_6 planes works, consider the matrices A and B below. The columns of A satisfy the equation $x_3 = x_4$, and the columns of B satisfy $x_1 = x_2$. Note that the corresponding 6-point planes have two points in common - the 2-point line ef.

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \begin{matrix} a' & b' & c' & d' & e & f \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

To find a matroid automorphism that maps G_1 to G_2, we let $x = [1, -1, 1, -1]$ and $y = [1, -1, -1, 1]$. Then $\{e, f, x, y\}$ is an orthoframe, i.e., $e, f \in P_x$ and $e, f \in P_y$. Reflection through the plane P_x is accomplished by $v \mapsto v - \frac{2v \cdot x}{x \cdot x} x$. This maps $a \mapsto d', b \mapsto c', c \mapsto b', d \mapsto a'$. The reader can check reflection through P_y maps $a \mapsto c', b \mapsto d', c \mapsto a', d \mapsto b'$. In either case, we have a map interchanging G_1 and G_2.

Alternatively, we can map one plane to the other by performing two row swaps on the matrix H: (13)(24). This is an *even* permutation of the rows, and so maps $M(H_4)$ to itself.

It is interesting to note that although $\mathrm{Aut}(M(H_4))$ acts transitively on pairs of intersecting 5-point lines, it does not act transitively on pairs of intersecting 3-point lines. The latter fall into two equivalence classes, as we have already seen: A pair of intersecting 3-point lines determines either a Π_6 or a Π_{15}.

$\mathrm{Aut}(M(H_4))$ also acts transitively on orthoframes. We omit the short proof.

Lemma 4.3. $\mathrm{Aut}(M(H_4))$ *acts transitively on orthoframes.*

Recall a group G acting on a set X is *primitive* if G acts transitively and preserves no non-trivial blocks of X. We now prove the action of $\mathrm{Aut}(M(H_4))$ on the points of the matroid is primitive.

Theorem 4.4. *The automorphism group action is primitive on the 60 points of the ground set of* $M(H_4)$.

Proof. Suppose E is partitioned into blocks, and suppose Δ is a block. Then, for any $\sigma \in \mathrm{Aut}(M(H_4)), \Delta \cap \sigma(\Delta) = \Delta$ or $\emptyset$. We must prove $|\Delta| = 1$ or 60.

Suppose $x \in \Delta$. Note for all $\sigma \in \mathrm{stab}(x)$, we must have $\sigma(\Delta) = \Delta$. Since $\mathrm{Aut}(M(H_3)) \cong S_5 \leq \mathrm{stab}(x)$ acts transitively on the 15 points of P_x, we must have either $P_x \subseteq \Delta$ or $P_x \cap \Delta = \emptyset$. There are now two cases to consider.

- If $P_x \subseteq \Delta$, then since P_x meets every other 15-point plane (from Prop. 3.3), we get $\sigma(P_x) \cap P_x \neq \emptyset$ for all $\sigma \in \mathrm{Aut}(M(H_4))$. Thus, $\Delta \cap \sigma(\Delta) = \Delta$ for all $\sigma \in \mathrm{Aut}(M(H_4))$, i.e., $\sigma(\Delta) = \Delta$ for all σ. But this immediately gives $\Delta = E$, i.e., Δ is the trivial block formed by the entire ground set of the matroid.

- If $P_x \cap \Delta = \emptyset$, we restrict to stab$(x)$ and consider all the lines that contain x. We know x is in 15 2-point lines, but the 15 points that produce these 2-point lines form P_x, so none of these 15 points is in Δ.

  There are ten 3-point lines through x, which we denote $\{L_1, L_2, \ldots, L_{10}\}$, as in the proof of Lemma 4.1. From that proof and the fact that the action of Aut$(M(H_4))$ is transitive on 3-point lines (Lemma 4.2), we must have either $L_i \subseteq \Delta$ for all $1 \leq i \leq 10$, or $\Delta \cap L_i = \{x\}$ for all i. (Note: Every $\sigma \in$ stab(x) maps the pencil of lines through x to itself, so each line contributes the same number of points to Δ, and reflecting through the plane P_x forces us to take 0 or 2 points from each L_i, not counting x.) Thus, Δ contains either 0 points or 20 points from the L_i pencil, not counting x.

  Using an analogous argument on the pencil of six 5-point lines through x, we find each such line must meet Δ in the same number of points, and that number must be 0, 2 or 4 per line (not counting x). This means Δ contains 0, 12 or 24 points from this pencil, again not counting x.

  Putting all of this together gives the $|\Delta| = 1, 13, 21, 25, 33$ or 45. But $|\Delta|$ must divide 60, since the blocks partition E. Thus $|\Delta| = 1$, so $\Delta = \{x\}$.

□

The next result follows immediately from Theorem 1.7 of [1].

Corollary 4.5. stab(x) *is a maximal subgroup of* Aut$(M(H_4))$.

It is worth pointing out that the action of Aut$(M(H_3))$ on $M(H_3)$ is imprimitive – the partition into rank-3 orthoframes is a non-trivial partition of the 15 elements of the matroid into 5 blocks. This corresponds geometrically to permuting the 5 cubes embedded in a dodecahedron.

The root system H_4 has a Coxeter/Weyl group of size 14,400. Coxeter's notation [3] for the group $[3, 3, 5]$ suggests its construction as a reflection group.

$$[3, 3, 5] = \langle R_1, R_2, R_3, R_4 \mid (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_4)^5 = I\rangle$$

In this presentation, we assume each R_i is a reflection, i.e., $R_i^2 = I$, and that $(R_iR_j)^2 = I$ for $|i - j| > 1$, i.e., reflections R_i and R_j are orthogonal for $|i - j| > 1$.

Conway and Smith (Table 4.3 of [2]) express this group as $\pm[I \times I] \cdot 2$, where $I \cong A_5$ is the chiral (or direct) symmetry group of the icosahedron. In 4-dimensions, $I \times I$ is best understood as a rotation group via quaternion multiplication.

Theorem 4.6. *Let W be the Coxeter/Weyl isometry group for the root system H_4, with center Z generated by central inversion $\mathbf{v} \mapsto -\mathbf{v}$.*

(1) $|\operatorname{Aut}(M(H_4))| = |W| = 14,400$.
(2) W/Z *is an index* 2 *subgroup of* $\operatorname{Aut}(M(H_4))$.

Proof. (1) From Lemma 4.1, we have $\operatorname{stab}(x) \cong S_5 \times \mathbb{Z}_2$. Since the orbit of x is all of E (as the automorphism group is transitive), we have $|\operatorname{Aut}(M(H_4))| = |S_5 \times \mathbb{Z}_2| \cdot |E| = 14,400$.
(2) Every isometry of W gives a matroid automorphism, and central inversion in W corresponds to the identity in $\operatorname{Aut}(M(H_4))$. The result now follows from (1).

□

In [4], $\operatorname{Aut}(M(H_4))$ is obtained as follows: First extend the root system H_4 by adding an isomorphic copy H_4' of H_4. Then $\operatorname{Aut}(M(H_4)) \cong W(H_4 \cup H_4')/Z$, where $Z \cong \mathbb{Z}_2$ is the subgroup generated by central inversion (Z is the center of W). The H_4' copy is obtained by using the field automorphism $\phi : \mathbb{Q}[\tau] \to \mathbb{Q}[\tau]$ given by $\tau \mapsto \bar{\tau}$ on the original root system H_4. (Note that this map must operate on a different set of coordinates than those treated here, since the 24 roots whose coordinates avoid τ are fixed by this map.)

We summarize this section with the following consequence of Theorem 4.6:

> The automorphism groups of the root systems H_3 and H_4 have the same connection to the Coxeter/Weyl groups $W(H_3)$ and $W(H_4)$. In each case, half of the matroid automorphisms are geometric and half are not. The non-geometric automorphisms arise from the S_5 action in $\operatorname{stab}(x)$ that permit odd permutations of rank-3 orthoframes in Π_{15} planes.

5. Geometric interpretations of $M(H_4)$.

We can interpret the flats and orthoframes of $M(H_4)$ in terms of the 120-cell and its dual, the 600-cell. We give the number of vertices, edges, 2-dimensional faces and 3-dimensional faces for the 120- and 600-cell in Table 3 - this information appears in Table 1(ii) of [3].

Table 3. Number of elements of the 120- and 600-cell.

Object	Vertices	Edges	2D faces	3D facets
120-cell	600	1200	720	120
600-cell	120	720	1200	600

The 2-dimensional faces of the 120-cell are pentagons and the 3-dimensional facets are dodecahedra; for the 600-cell, 2-dimensional faces are triangles and 3-dimensional facets are tetrahedra.

Now each of the 60 points of the matroid corresponds to a pair of roots $\pm\mathbf{v}$ of the root system H_4. Since the roots are also the vertices of the 600-cell, we immediately get a correspondence between the points of the matroid and the pairs of opposite vertices of the 600-cell. We can interpret the other geometric elements of the 120- and 600-cell through the matroid $M(H_4)$ in Table 4. We also remark the 75 matroid orthoframes correspond to 75 embedded hypercubes in the 120-cell or 600-cell.

TABLE 4. Correspondence between geometric elements and flats in $M(H_4)$.

Geometric Family		Matroid Flat
120 Vertices of 600-cell	$\Leftrightarrow$	60 Points
1200 Triangles of 600-cell	$\Leftrightarrow$	600 Π_3's
720 Pentagons of 120-cell	$\Leftrightarrow$	360 Π_5's
600 Tetrahedra of 600-cell	$\Leftrightarrow$	300 Π_6's
120 Dodecahedra of 120-cell	$\Leftrightarrow$	60 Π_{15}'s

We comment briefly on some of these connections. For the root system H_3, this correspondence is explored in detail in [5]. In that case, the roots are parallel to the edges of an icosahderon. This makes the matroid correspondence immediate: 3-point lines of the matroid correspond to pairs of triangles in the icosahedron and 5-point lines correspond to pairs of vertices of the icosahedron (or pentagons of the dual dodecahedron).

The chief difficulty in applying the results of [5] to H_4 arises from the fact that the edges of the 120-cell or 600-cell are no longer parallel to the roots. But, since each 15-point plane is isomorphic to $M(H_3)$ as a matroid, the correspondence between dodecahedra and Π_{15}'s is clear. We explain the connection between the 720 pentagons in the 120-cell and the 360 Π_5's in the matroid. In the 120-cell, a given pentagon is in two dodecahedra, but in $M(H_4)$, a given 5-point line is in 5 Π_{15}'s. We can "correct" this by using Π_5's, since each 5-point line of the matroid is in precisely five Π_5's.

Finally, we can use the orthopoint-orthoplane bijection to get a matroidal interpretation for the 120-cell/600-cell duality.

Proposition 5.1. *Let F be the collection of 15-point planes, and let B be the bipartite graph with vertex set $E \cup F$ with an edge joining the point x to the plane P if and only if $x \in P$. Then* $\mathrm{Aut}(B) \cong \mathrm{Aut}(M(H_4)) \times \mathbb{Z}_2$.

Proof. It is clear the bipartite graph B allows us to reconstruct all the flats of the matroid, and any matroid automorphism acting on E will necessarily be a graph automorphism of B. Further, we can swap the points and the planes – map a point x to its orthoplane P_x.

□

We conclude by observing that it should be possible to treat matroids associated to other root systems (especially the exceptional E_6, E_7, and E_8) in a coherent way that also explains the structure of those matroids. We hope to undertake such a program in the future.

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