CHROMATIC AND TUTTE POLYNOMIALS FOR GRAPHS, ROOTED GRAPHS AND TREES

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ABSTRACT. The chromatic polynomial of a graph is a one-variable polynomial that counts the number of ways the vertices of a graph can be properly colored. It was invented in 1912 by G.D. Birkhoff in his unsuccessful attempt to solve the four-color problem. In the 1940's, Tutte generalized Birkhoff's polynomial by adding another variable and analyzing its combinatorial properties. The Tutte polynomial itself has been generalized to other combinatorial objects, with connections to knot theory, state changes in physics, probability and other areas. We concentrate on an extension to rooted trees where the polynomial is a complete invariant, i.e., two rooted trees T_1 and T_2 are isomorphic iff $f(T_1) = f(T_2)$.

1. HISTORICAL INTRODUCTION

How many colors are needed to color a map so that regions sharing a border receive different colors? This problem traces its origin to Oct. 23, 1852, when Francis Guthrie asked his professor, Augustus De Morgan, whether he knew of a solution. DeMorgan, in turn, asked his friend, Sir William Rowan Hamilton, who was also stumped. This innocent question raised by a student inspired an enormous amount of research in graph theory, with a final resolution of the question occurring more than 100 years later.

It's easy to see that at least four colors are necessary. For instance, in South America, note that the four countries Argentina, Brazil, Bolivia and Paraguay must all receive different colors since each pair of these countries shares a border – see Fig. 1. To prove that four colors also suffice for maps in the plane (or, equivalently, on a sphere) occupied some of the best mathematicians of the 19^{th} and 20^{th} centuries.

Instead of coloring the regions of maps, most mathematicians prefer to color the vertices of graphs. A *proper* vertex coloring of a graph G is an assignment of colors to the vertices of G so that adjacent vertices receive different colors. Converting a map to a graph is straightforward: Each region of the map becomes a vertex of the graph, and two vertices of the graph are joined by an edge precisely when the corresponding regions share a boundary.

Guthrie's problem can now be stated in graph-theoretic terms:

Conjecture 1.1. *Four Color Conjecture: Every planar graph can be properly colored using at most four colors.*

A very quick history of the progress on the conjecture is given below, culminating in the famous proof of Appel and Haken in 1976 that made extensive use of computers. This proved to be a source of controversy at the time, and many

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FIGURE 1. A famous continent.

mathematicians are still not completely satisfied with the nature of this proof. (A shorter proof that still requires some computer checking was found by [17].) See [20] for a fairly extensive recounting of the colorful history of this problem.

- 1852 - Problem posed by Guthrie.
- 1871 - De Morgan dies.
- 1878 - Arthur Cayley revives interest in the problem.
- 1879 - Alfred Kempe 'solves' the problem.
- 1889 - Percy Heawood finds a flaw in Kempe's proof.
- 1912 - G. D. Birkhoff introduces the chromatic polynomial.
- 1976 - Appel and Haken publish 'correct' proof.

This paper is organized as follows. In Section 2, we define the chromatic polynomial and develop its basic properties. Section does the same for the Tutte polynomial, including the connections between the Tutte polynomial and three one-variable polynomials, the chromatic, flow and reliability polynomials.

In Section 4, we extend the definition of the Tutte polynomial to *rooted* graphs, i.e., graphs with a distinguished vertex, concentrating especially on rooted trees. Section 5 explores the difference between the ordinary Tutte polynomial and the rooted version. In sharp contrast to the situation for ordinary trees and the (ordinary) Tutte polynomial, we highlight a Theorem from [11]:

Theorem 5.1 Let T_1 and T_2 be rooted trees. Then $f(T_1; x, y) = f(T_2; x, y)$ iff T_1 and T_2 are isomorphic.

If T is an ordinary tree, then it is possible to define two different Tutte polynomials, one based on cycle-rank (the ordinary Tutte polynomial) and one based on a *greedoid* rank function. The greedoid version is much sharper at distinguishing different trees, but the result analogous to the rooted tree result (Theorem 5.1) is false (Example 5.6):

Example 5.6 There are non-isomorphic trees with the same greedoid Tutte polynomials [6].

It is possible to give combinatorial interpretations to the two greedoid-based Tutte polynomials defined in Section 4 and 5. In each case, we can formulate the Tutte polynomial in purely graph-theoretic terms., allowing us to obtain tree reconstruction results (Corollary 5.8).

We conclude with some open problems in Section 6. We also thank John Kennedy for his encouragement and interest.

2. The chromatic polynomial of a graph

In 1912, George Birkhoff introduced a polynomial, the *chromatic polynomial* to count the number of proper colorings of a graph. More precisely, let $\chi(G;\lambda)$ be the number of proper colorings of a graph G using λ (or fewer) colors.

Example 2.1. Let G be the graph obtained by removing an edge from K_4, as in Fig. 2.

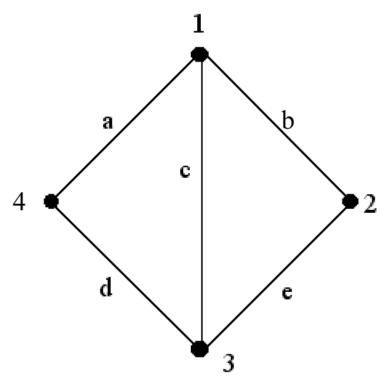

Figure 2. $\chi(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2$.

In this case, we can determine the chromatic polynomial 'greedily.' There are λ colors available for vertex 1, $\lambda - 1$ colors left for vertex 2, and $\lambda - 2$ colors for vertex 3. Now there are also $\lambda - 2$ colors available for vertex 4, so we get $\chi(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2$.

Note that $\chi(G;3) = 6 > 0$, so it is possible to properly color the vertices of G using 3 colors. Birkhoff hoped that studying the roots of $\chi(G;\lambda)$ could lead to a proof of the four-color theorem.

Theorem 2.2 (Four Color Theorem)**.** *If G is a planar graph, then $\chi(G;4) > 0$.*

Computing $\chi(G;\lambda)$ for an arbitrary graph is difficult. In fact, determining if the chromatic number of a graph equals k (for $k \geq 3$) is an NP-complete problem. This was one of the first problems to be shown to be NP-complete in 1972 – see [7]. More on the complexity of computing chromatic invariants can be found in [4].

$\chi(G;\lambda)$ satisfies an important recursive formula that allows for inductive proofs. Recall that if G is a graph and e is an edge of G, then the *deletion* $G - e$ is formed by simply removing the edge e from G. The *contraction* G/e of the (non-loop) edge e is obtained from G by identifying the two endpoints of e and then removing e. Thus, if G has n edges, then $G - e$ and G/e each have $n - 1$ edges.

Suppose u and v are the two endpoints of the edge e and partition the proper colorings of $G - e$ as follows: those in which u and v receive different colors, and those in which they receive the same colors. In the former case, we get a proper coloring of G; in the latter, we get a proper coloring of G/e. Further, all proper colorings of G and G/e arise in this way. This proves the next result.

Theorem 2.3. *[Deletion-Contraction] Let G be a graph and e be a non-loop edge. Then*

$$\chi(G;\lambda) = \chi(G - e;\lambda) - \chi(G/e;\lambda).$$

It follows from Theorem 4.2 and induction that $\chi(G;\lambda)$ is actually a *polynomial.* Note that this is not immediate from Birkhoff's definition.

Corollary 2.4. *$\chi(G;\lambda)$ is a polynomial in λ.*

Example 2.5. As an example of the deletion-contraction method, we again compute $\chi(G;\lambda)$ for G of Fig. 2. In Fig. 3, we show the result of deleting and contracting the edge c. Note that $G - c$ is isomorphic to C_4, the 4-cycle, and G/c is isomorphic to a join of two 2-cycles. One can check $\chi(G/c,\lambda) = \lambda(\lambda - 1)^2$, while $\chi(G - c,\lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)$. This agrees with our previous calculation: $\chi(G;\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$.

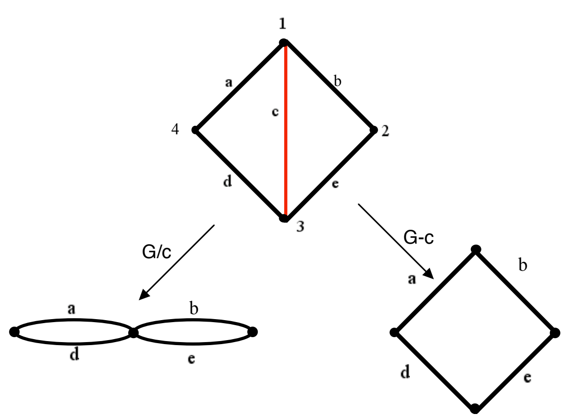

Figure 3. Deletion-Contraction

One can use the deletion-contraction formula to establish other results by induction. For instance, the coefficients of $\chi(G;\lambda)$ alternate in sign. More information about the chromatic polynomial and some of its elementary properties can be found in [16] or [5], for instance.

We will need the next result, which is a standard exercise.

Proposition 2.6. *Let T be a tree with n edges. Then $\chi(T;\lambda) = \lambda(\lambda-1)^{n-1}$.*

Studying the roots of $\chi(G;\lambda)$ is still a very active area of research. For instance, a somewhat surprising application to statistical mechanics is explained in [18], where the chromatic polynomial models state changes.

3. The Tutte polynomial of a graph

William Tutte was one of the giants of graph theory and combinatorics in the 20^{th} century. His work at Bletchley Park as a codebreaker has been called "one of the greatest intellectual feats of World War II."

While working on a recreational problem involving the partition of a square into squares of distinct sizes (the 'squared square' problem), he noticed that the number of spanning trees $n(G)$ in a connected graph obeys a deletion-contraction recursion: $n(G) = n(G-e) + n(G/e)$ (provided e is neither an isthmus nor a loop). He investigated other invariants that satisfied similar recursive formulas, leading to the following definition.

Definition 3.1. Let G be a graph on the edge set E. The *Tutte polynomial* $f(G;x,y)$ is defined as follows:

(1) If e is neither an isthmus nor a loop, then
$$f(G;x,y) = f(G/e;x,y) + f(G-e;x,y)$$
(2) If e is an isthmus, then
$$f(G;x,y) = x \cdot f(G/e;x,y)$$
(3) If e is a loop, then
$$f(G;x,y) = y \cdot f(G-e;x,y)$$

In order to ensure $f(G;x,y)$ is well-defined, we must make sure the polynomial obtained from repeated deletions and contractions does not depend on the order we operate on the edges. The following theorem establishes this.

Theorem 3.2. *Let G be a graph with edge set E. For $A \subseteq E$, let $r(A)$ be the size of the largest cycle-free subset of A. Then*
$$f(G;x,y) = \sum_{A\subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Proving Theorem 3.2 is straightforward using induction. The expression $r(A)$ is usually called the *cycle rank* of A, and it allows us to extend the definition of the Tutte polynomial to any objects possessing a rank function. In particular, this invariant has been extended to matroids and greedoids; see [2]. Borrowing terminology from linear algebra, the expression $r(E) - r(A)$ is called the *corank* of A and $|A| - r(A)$ is the *nullity* of A.

The reader is encouraged to use either 3.1 or 3.2 to establish the following.

Proposition 3.3. (1) *Let T be a tree with n edges. Then*
$$f(T;x,y) = x^n.$$

(2) *Let C_n be a cycle with n edges. Then*

$$f(C_n;x,y) = x^{n-1} + x^{n-2} + \cdots + x + y.$$

Example 3.4. As an example, we compute the Tutte polynomial $f(G;x,y)$ for the graph G of Fig. 2. Note that the deletion-contraction definition allows us to use the decomposition given in Fig. 3. Thus, $f(G) = f(G-c) + f(G/c) = f(C_2)^2 + f(C_4)$. This uses the fact that $f(G_1 \oplus G_2) = f(G_1)f(G_2)$, where $G_1 \oplus G_2$ is the graph obtained from G_1 and G_2 by identifying a vertex of G_1 with a vertex of G_2. Now we can use 3.3(2) to get $f(G;x,y) = x^3 + 2x^2 + 2xy + x + y + y^2$.

The Tutte polynomial has a certain universal property: Any invariant satisfying a deletion-contraction recursion is (essentially) an evaluation of the Tutte polynomial. This is made precise by a theorem of Brylwaski – see 6.2.2 of [2]. For now, we list several important evaluations of $f(G;x,y)$.

Theorem 3.5. *Let G be a graph.*

(1) *Subsets: The number of subsets of edges is $f(G;2,2)$.*
(2) *Spanning trees: The number of spanning trees is $f(G;1,1)$.*
(3) *Spanning sets: The number of subsets of edges that contain a spanning tree is $f(G;1,2)$.*
(4) *Acyclic subsets of edges: The number of subsets of edges contained in a spanning tree is $f(G;2,1)$.*
(5) *Acyclic orientations: The number of acyclic orientations of the edges is $f(G;2,0)$.*
(6) *Acyclic orientations with a unique specified source: The number of acyclic orientations of the edges in which a specified vertex v is the unique source is $f(G;1,0)$ (and, in particular, does not depend on the vertex v chosen as the source).*
(7) *Totally cyclic orientations: The number of orientations of G in which every edge is in some cycle is $f(G;0,2)$.*
(8) *Distinct score vectors: The number of distinct score vectors that arise from orientations of the edges is $f(G;2,1)$.*

Note that the evaluation $f(G;2,1)$ counts two distinct invariants; both the number of acyclic sets and the number of score vectors. Property (5) is due to Stanley [19], while (6) and (7) are proven in [15]. We also draw the reader's attention to property (2), the prototypical evaluation Tutte noticed in the 1940's. Tutte was able to use *basis activities* to set up a bijection between spanning trees and individual terms of the polynomial. His work is valid in the more general context of matroids; see 6.6.A of [2] for more.

In addition to these numerical invariants, the Tutte polynomial also encodes three important one-variable graph polynomials. We now define the flow polynomial of a graph. A λ-flow is obtained by choosing an orientation of the edges of G and assigning an element of the additive group $\mathbb{Z}_\lambda$ to each edge so that Kirchoff's law is satisfied at each vertex (i.e., the sum of the weights of the edges directed toward v equals the sum of the weights of the edges directed away from v). A λ-flow is *nowhere zero* if no edge is assigned 0.

Definition 3.6. The flow polynomial: $\chi^*(G;\lambda)$ is the number of nowhere zero λ-flows of the graph G.

It is worth pointing out two features of this polynomial:

- $\chi^*(G;\lambda)$ does not depend on the initial orientation chosen for the edges of G: reversing the direction of an edge of weight x is equivalent to replacing its weight with $-x$.
- $\chi^*(G;\lambda)$ does not depend on the abeilan group of order λ used in the definition. We chose $\mathbb{Z}_\lambda$, but any abelian group of order λ would work equally well.

Our last one-variable invariant is the *reliability polynomial*. This polynomial is treated in depth in [4].

Definition 3.7. Let G be a graph and suppose each edge is independently operational with probability p. The *reliability polynomial* $R(G;p)$ is the probability that the number of components of G does not increase.

The chromatic, flow and reliability polynomials can all be found from certain evaluations of the Tutte polynomial.

Theorem 3.8. *Let G be a graph with m vertices, n edges and c connected components.*

(1) $\chi(G;\lambda) = \lambda^c(-1)^{m-c} f(G;1-\lambda,0)$.
(2) $\chi^*(G;\lambda) = (-1)^{n-m+c} f(G;0,1-\lambda)$.
(3) $R(G;p) = (1-p)^{n-m+c}p^{m-c} f(G;0,\frac{1}{1-p})$

Proofs of these results follow from applying deletion and contraction and using induction. See Propositions 6.3.1, 6.3.4 and Example 6.2.7 of [2]. The close connection between the evaluations giving the chromatic and flow polynomials has an interpretation via duality.

We conclude this section with a fundamental result concerning duality which extends to matroids.

Theorem 3.9. *Let G be a planar graph with planar dual G^*. Then*

$$f(G^*;x,y) = f(G;y,x).$$

Thus, the flow polynomial carries information about coloring the dual graph when G is planar. When G is a planar graph, one can then view the Tutte polynomial as simultaneously carrying information about coloring a graph and its dual. In fact, Tutte called (a version of) $f(G;x,y)$ the *dichromate* for this reason.

4. The Tutte polynomial of a rooted graph

Rooted graphs are simply graphs with a distinguished vertex. Such graphs are important in many applications, especially in communication theory in which one vertex plays a special role (a server, for instance). Thus, it is worthwhile to search for an extension of the Tutte polynomial to rooted graphs.

We now define the Tutte polynomial of a rooted graph; this definition follows Theorem 3.2.

Definition 4.1. Let G_v be a rooted graph with edge set E and root vertex v. For $A \subseteq E$, define the rank $r(A)$ via subtrees rooted at v:

$$r(A) = \max_{F \subseteq A}\{|F| : F \text{ is a rooted subtree of } T\}.$$

Then the *Tutte* polynomial $f(G_v; x, y)$ is defined by

$$f(G_v; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

$r(A)$ is the *branching rank* of the *greedoid* associated to the edge set of the rooted graph. Greedoids generalize matroids, which can be thought of as a simultaneous generalization of graphs and finite subsets of a vector space. Greedoids were introduced by Korte and Lovasz in a series of papers in the 1980's; see [1] for some general background in the subject. For our purposes, we will not need this level of generality.

Given the variety of evaluations (Theorem 3.5) and the very important deletion-contraction property (Definition 3.1) the (ordinary) Tutte polynomial satisfies, it is reasonable to try to extend as many of these results as possible to the rooted case.

We write $r(G_v)$ for $r(E)$, $r(G_v - e)$ for $r(E - e)$ when computed in the rooted graph $G_v - e$, and so on.

Theorem 4.2. *Let G_v be a rooted graph on n edges and let e be an edge adjacent to v. Then*

$$f(G_v; x, y) = f(G_v/e; x, y) + (x-1)^{r(G_v)-r(G_v-e)} f(G_v - e; x, y).$$

If there is no edge adjacent to the root, then $f(G_v; x, y) = y^n$.

Theorem 4.2 is proven in [11]. The term $(x-1)^{r(G_v)-r(G_v-e)}$ does not appear in the recursion for ordinary graphs (or matroids, for that matter) because $r(G) = r(G - e)$ provided e is not an isthmus. (An alternative approach is taken in [1], where a one-variable greedoid Tutte polynomial is defined. That polynomial can be obtained from the two-variable polynomial defined here by setting $x = 1$. This allows the deletion-contraction recursion to match that given in Definition 3.1, but the one-variable polynomial carries much less information than the two-variable version.)

Example 4.3. We compute $f(G_1; x, y)$ for the graph G given in Fig. 2, rooted at vertex 1. We will need the following formula for rooted cycles. The proof of 4.4 follows from Theorem 4.2 and induction.

Proposition 4.4. *Let C_n be a cycle on n edges, where the root can be placed at any vertex. Then*

$$f(C_n; x, y) = n - 1 + y + \sum_{k=1}^{n-1} (n-k)(x-1)^k y^{k-1}.$$

To find $f(T_1; x, y)$, we use deletion-contraction from Theorem 4.2. As before, we get $f(G_1) = f(G_1 - c) + f(G_1/c) = f(C_2)^2 + f(C_4)$. This time, using the formula from 4.4, we get

$$f(G_1; x, y) = 3x + x^2 + 3y - 2xy + 2x^2y + 3xy^2 - 3x^2y^2 + x^3y^2.$$

Unlike the ordinary Tutte polynomial, $f(G_v; x, y)$ can have negative coefficients. While it is still true that $f(G_v; 1, 1)$ counts the number of spanning trees of G_v, it is not possible to set up a direct bijection between the spanning trees and the terms of the polynomial. In spite of the problem with negative coefficients, the rooted Tutte polynomial shares many of the same evaluations the ordinary polynomial possesses.

Proposition 4.5. *Let G_v be a rooted graph.*

(1) *Subsets: The number of subsets of edges is $f(G_v; 2, 2)$.*
(2) *Spanning trees: The number of spanning trees is $f(G_v; 1, 1)$.*
(3) *Spanning sets: The number of subsets of edges that contain a spanning tree is $f(G_v; 1, 2)$.*
(4) *Rooted subtrees: The number of rooted subtrees is $f(G_v; 2, 1)$.*
(5) *Acyclic orientations with unique source v: The number of acyclic orientations of the edges in which v is the unique source is $f(G; 1, 0)$.*

Properties (1)-(4) are proven in much the same way as before. Property (5) is also straightforward using induction [13]. Connections between the different expansions of the ordinary Tutte polynomial are developed in [14]. For rooted graphs, it is still possible to develop a theory of activities; see [12].

5. Chromatic and Tutte uniqueness

A class of graphs $\mathcal{C}$ is said to be *Tutte unique* if any two graphs in $\mathcal{C}$ have different Tutte polynomials. A simple example of a Tutte-unique class of graphs is the class consisting of all cycles.

In the opposite direction, it is easy to find two graphs with the same Tutte polynomial: Any tree on n edges has $f(T; x, y) = x^n$ (Proposition 3.3(1)). It follows from Theorem 3.8(1) (or Proposition 2.6) that the chromatic polynomial also fails to distinguish two tees on the same number of edges. Even if we ignore trees (which are not 2-connected), it is hopeless to expect the Tutte polynomial to distinguish all graphs; there are simply more graphs than there are potential Tutte polynomials. It follows from the pigeonhole principle that for any positive integer N, there are N non-isomorphic 2-connected graphs all having the same Tutte polynomial. See Exercise 6.9 of [2] for the matroid version of this argument.

5.1. Rooted trees.

The situation for rooted trees and the Tutte polynomial is much different. In this case, using the branching rank function, we get the following.

Theorem 5.1. *Let T_1 and T_2 be rooted trees. Then $f(T_1; x, y) = f(T_2; x, y)$ iff T_1 and T_2 are isomorphic.*

This is the main result of [11]. We give a sketch of the proof, which follows from two lemmas, both of which are proven in [11].

Lemma 5.2. *Let G_1 and G_2 be two disjoint rooted graphs, rooted at vertices v_1 and v_2, resp. Let $G_1 \oplus G_2$ be the rooted graph formed by gluing G_1 and G_2 together with new root vertex created by identifying the two roots v_1 and v_2. Then*

$$f(G_1 \oplus G_2) = f(G_1) f(G_2)$$

.

Lemma 5.3. *Let T_v be a tree and suppose the root vertex v has degree 1. Then $f(T_v; x, y)$ is an irreducible polynomial in the polynomial ring $\mathbb{Z}[x, y]$.*

Sketch of proof of 5.1 We show how to reconstruct the rooted tree T_v from the polynomial $f(T_v; x, y)$. The proof uses the two lemmas and math induction.

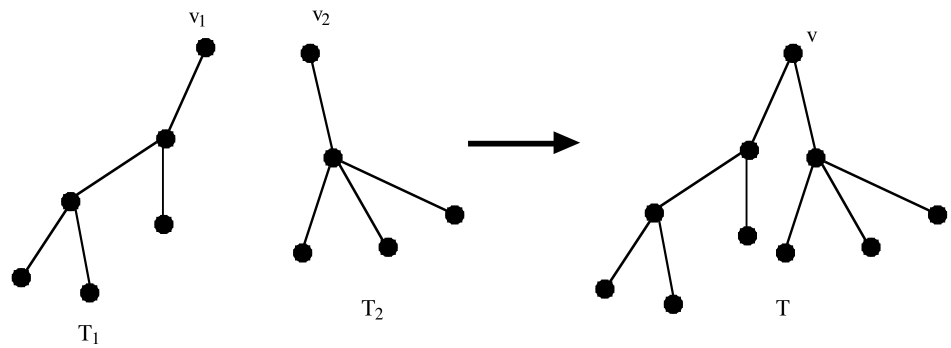

FIGURE 4. Tree identification

- **Case 1**: The polynomial $f(T_v; x, y)$ factors in a non-trivial way. Then each irreducible factor of $f(T_v)$ must correspond to a rooted subtree in which the degree of the root is 1 (this follows from 5.2 and 5.3). Now reconstruct each factor by induction, then glue them all together by identifying all of the roots. See Fig. 4.
- **Case 2**: $f(T_v; x, y)$ does not factor. Then the root vertex v must have degree 1 (again from 5.2 and 5.3). Let e be the edge incident to v and suppose the highest power of x appearing in $f(T_v; x, y)$ is n. Then $f(T_v/e; x, y) = f(T_v; x, y) - (x-1)^n y^{n-1}$. By induction, we can now reconstruct T_v/e, so we can reconstruct T_v by adding the edge e.

5.2. **Unrooted trees.** When T is an ordinary, unrooted tree, the ordinary Tutte polynomial is well-defined. Unfortunately, $f(T) = x^n$ gives no information about the structure of T. We now give a finer invariant for ordinary trees. We first need to define the rank of a subset of edges.

Definition 5.4. Let T be an unrooted tree with edge set E, with $|E| = n$. For $A \subset E$, define the *rank* of A to be the size of the largest subtree complement contained in A:

$$r(A) = \max_{F \subseteq A} \{|F| : E - F \text{ is a subtree}\}.$$

It's probably easier to think of $r(A)$ algorithmically. Given a subset A, we repeatedly *prune* the leaves of A: More precisely, given $A \subseteq E$, remove all leaves in A (*leaves* are edges with a vertex of degree 1), then remove all edges of A that became leaves after you removed the first batch of leaves, and so on. Let F be the collection of all edges of A that were removed at some stage of this process. Then $r(A) = |F|$. F is called a *feasible set.*

This rank function is usually called the *pruning* rank and it gives T an *antimatroid* structure. As usual, we won't need that level of generallity. Antimatroids are an important class of greedoids with closed sets satisfying a certain *anit-exchange* condition. See [1] for more on antimatroids and their relationship to greedoids.

This definition of rank will enable us to define a Tutte polynomial $g(T; x, y)$ exactly as before:

$$g(T; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

In light of Theorem 5.1, it is reasonable to make the following conjecture:

Conjecture 5.5. *Let T_1 and T_2 be unrooted trees. Then $g(T_1; x, y) = g(T_2; x, y)$ iff T_1 and T_2 are isomorphic.*

This conjecture is false. We give a counterexample from [6].

Example 5.6. Let T_1 and T_2 be the two trees of Fig. 5. Note that each tree has 10 edges, so the definition of $g(T)$ requires us to compute the rank of all 2^{10} subsets of edges of each tree. We postpone the computation of $g(T_i)$ until we give a combinatorial interpretation, when we have a better way to express $g(T)$. We promise to return to this.

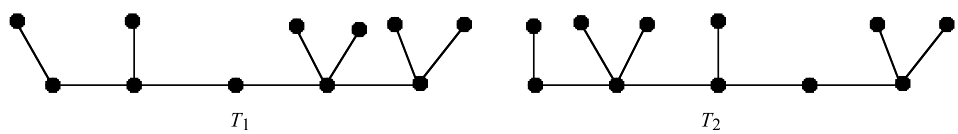

FIGURE 5. $g(T_1) = g(T_2)$.

T_1 and T_2 are examples of *caterpillars*, i.e., trees in which all edges are adjacent to a single path. The central path is called the *spine* of the caterpillar, even though caterpillars are invertebrates.

5.3. **Combinatorial interpretations.** The Tutte polynomial of a rooted or unrooted tree can be expressed in purely graph-theoretic terms. The basic idea is to group the terms in the subset expansions of the polynomials so that we can sum over *subtrees* instead of subsets.

We will need the following notation. For a rooted tree T_v having n edges, we let $\mathcal{S}$ be the collection of all subtrees of T rooted at v. If $S \in \mathcal{S}$, then let $m(S)$ be the number of edges x that can be added to S so that $S \cup \{x\}$ is a rooted subtree. If T is unrooted, we let $\mathcal{S}$ be the collection of all subtrees of T, and if $S \in \mathcal{S}$, then let $i(S)$ be the number of *internal* edges of S, i.e., the edges of S that are not leaves.

Theorem 5.7 is proven in [3].

Theorem 5.7. (1) *Let T_v be a rooted tree. Then*

$$f(T_v; x, y) = \sum_{S \in \mathcal{S}} (x-1)^{n-|S|} y^{n-|S|-m(S)}.$$

(2) *For an unrooted tree T, we have*

$$g(T; x, y) = \sum_{S \in \mathcal{S}} (x-1)^{|S|} y^{i(S)}.$$

SKETCH OF PROOF. We give the idea for part (2). Let S be a subtree and consider all subsets $A \subseteq E$ with $F = E - S \subseteq A$ and $|F|$ maximum. For each such A, F is the *unique* subtree complement contained in A of maximum size. This follows because the union of two subtree complements is also a subtree complement (equivalently, the intersection of any two subtrees is a subtree). Then $r(A) = |F| = n - |S|$ for all these subsets A, so they all have $r(E) - r(A) = |S|$.

What can A look like? Using our idea of repeated pruning to determine $r(A)$, we note that A must contain $F = E - S$. What additional edges can be in A? Precisely those edges that cannot be pruned after F has been pruned, i.e., the *internal* edges of S. The binomial theorem allows us to group these terms together: $\sum(x-1)^{|S|}(y-1)^{|A|-|F|} = (x-1)^{|S|}y^{i(S)}$.

Example 5.6, continued. We now apply Theorem 5.7 to complete the computation of $g(T)$ from Example 5.6 (see Fig. 5), keeping our promise. Note that T_1 and T_2 have the same degree sequence. Since $i(T_1) = i(T_2) = 4$, we also have $i(S) \leq 4$ for any subtree of T_1 or T_2. In Table 1, we list the size of all the subtrees, together with the number of internal edges each subtree has. We omit entries in the table with no subtrees; for example, there are no subtrees with 8 edges and 2 internal edges. We also point out that the counts are facilitated by noticing that the only edges that can be internal in any subtree are the 4 edges along the spine of T_1 or T_2.

TABLE 1. Subtree data for T_1 and T_2 of Fig. 5.

$\|E(S)\|$	$i(S)$	Count	Term
0	0	1	1
1	0	10	$10(x-1)$
2	0	14	$14(x-1)^2$
3	0	6	$6(x-1)^3$
	1	13	$13(x-1)^3y$
4	0	1	$(x-1)^4$
	1	14	$14(x-1)^4y$
	2	9	$9(x-1)^4y^2$
5	1	6	$6(x-1)^5y$
	2	15	$15(x-1)^5y^2$
	3	7	$7(x-1)^5y^3$
6	1	1	$(x-1)^6y$
	2	9	$9(x-1)^6y^2$
	3	18	$18(x-1)^6y^3$
	4	2	$2(x-1)^6y^4$
7	2	2	$2(x-1)^7y^2$
	3	17	$17(x-1)^7y^3$
	4	7	$7(x-1)^7y^4$
8	3	7	$7(x-1)^8y^3$
	4	9	$9(x-1)^8y^4$
9	3	1	$(x-1)^9y^3$
	4	5	$5(x-1)^9y^4$
10	4	1	$(x-1)^{10}y^4$

Summing the final column gives the shared Tutte polynomial. For convenience, we set $t = x - 1$:

$$\begin{aligned} g(T_i; t+1, y) \quad = \quad & y^4t^{10} + 5y^4t^9 + y^3t^9 + 9y^4t^8 + 7y^3t^8 + 7y^4t^7 + 17y^3t^7 + 2y^2t^7 \\ & +2y^4t^6 + 18y^3t^6 + 9y^2t^6 + yt^6 + 7y^3t^5 + 15y^2t^5 + 6yt^5 \\ & +9y^2t^4 + 14yt^4 + t^4 + 13yt^3 + 6t^3 + 14t^2 + 10t + 1. \end{aligned}$$

We now restate Theorem 5.1 in combinatorial terms. We also include a related result obtained by using a pruning rank function for rooted trees [3]. For a rooted tree T_v, let $c_{i,l}$ be the number of subtrees S rooted at v with precisely i internal edges and l external edges. Let $d_{s,m}$ be the number of rooted subtrees S on s edges having exactly m edges e of $T_v - S$ with $S \cup \{e\}$ a rooted subtree.

Corollary 5.8. *Let T_v be a rooted tree.*

1. *T_v can be uniquely reconstructed from the sequence $\{c_{i,l}\}$.*
2. *T_v can be uniquely reconstructed from the sequence $\{d_{s,m}\}$.*

More colloquially, we can reconstruct a rooted tree from the knowledge of the number of subtrees with i internal and l external edges for all i and l.

The counterexample of Example 5.6 shows that the same is not true for unrooted trees. In purely combinatorial terms, we have:

Unrooted trees It is not possible in general to reconstruct a tree from the sequence $\{c_{i,l}\}$.

6. Open problems

1. In the spirit of the study of chromatic- and Tutte-uniqueness for ordinary graphs, it would be interesting to extend Theorem 5.1 to other classes of rooted graphs. For instance, is it true that any two rooted graphs on n edges and n vertices have distinct rooted Tutte polynomials? More generally, we propose the following:

   Problem 6.1. *Find a non-trivial class of rooted graphs $\mathcal{C}$ so that, for $G_1, G_2 \in \mathcal{C}$, we have $f(G_1) = f(G_2)$ iff $G_1 \cong G_2$.*

2. There are several interesting evaluations of the Tutte polynomial – see Theorem 3.5. In Proposition 4.5, we list a few evaluations of the rooted version of the Tutte polynomial. It should be possible to extend/interpret other evaluations of the ordinary Tutte polynomial in the rooted case.

   Problem 6.2. *Extend all parts of Theorem 3.5 to the rooted case.*

3. The chromatic polynomial is an evaluation of the ordinary Tutte polynomial (Theorem 3.8): $\chi(G;\lambda) = \lambda^c(-1)^{m-c}f(G;1-\lambda,0)$. This suggests that it should be possible to extend the definition of the chromatic polynomial to rooted graphs by applying the same evaluation to the rooted Tutte polynomial:

   $$\chi(G_v;\lambda) = \lambda^c(-1)^{m-c}f(G_v;1-\lambda,0).$$

   It should be worthwhile to study this polynomial and the combinatorial interpretations of its evaluations at positive integers. A similar comment applies to the flow polynomial.

(4) Although the Tutte polynomial of an unrooted tree of Section 5 does not uniquely determine the tree, there may be interesting classes of trees for which this data is a complete invariant. Some information in this direction appears in [10]. More precisely, we propose the following:

Problem 6.3. *Find a non-trivial class of trees $\mathcal{T}$ so that, for $T_1, T_2 \in \mathcal{T}$, we have $g(T_1) = g(T_2)$ iff $T_1 \cong T_2$.*

(5) It is possible to extend the Tutte polynomial for rooted trees to posets. This is the focus of [8] and [9]. In this context, it is not difficult to find combinatorial interpretations for the polynomial. It is also possible to find two posets with the same Tutte polynomial: $f(N) = f(3 \oplus 1)$. We conjecture that the Tutte polynomial is a complete invariant for *series-parallel posets*, however.

Conjecture 6.4. *If P_1 and P_2 are series-parallel posets with $f(P_1) = f(P_2)$, then $P_1 \cong P_2$.*

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