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Algebraic Characteristic Sets of Matroids

GARY GORDON

Department of Mathematics, Lafayette College, Easton, Pennsylvania 18042

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For a matroid M, define the algebraic characteristic set $\chi_A(M)$ to be the set of field characteristics over which M can be algebraically represented. We construct many examples of rank three matroids with finite, non-singleton algebraic characteristic sets. We also determine $\chi_A(PG(2,p))$ and $\chi_A(AG(2,p))$. An infinite family of rank three matroids with empty algebraic characteristic set is constructed. In addition, we answer some antichain and excluded minor questions for algebraic representability over a given field F. © 1988 Academic Press, Inc.

1. INTRODUCTION

The theory of algebraic matroids has received relatively little attention compared with many other areas of matroid theory. Ingleton and Main produced the first example of a non-algebraic matroid in 1975 [5] and more recently, Lindström has obtained results concerning algebraic matroids. The main purpose of this paper is to prove that for certain matroids with finite, non-empty linear characteristic sets, the algebraic and linear characteristic sets agree.

We assume familiarity with the basic definitions of matroid theory. The background material can be found in [3] or [13], for example. We now remind the reader of some definitions.

DEFINITION. A matroid M is *algebraic* over a field F if there is a mapping $f\colon M \to E$, E an extension field of F, such that $S \subseteq M$ is independent iff $|f(S)| = |S|$ and $f(S)$ is algebraically independent over F. Define the *algebraic characteristic set*, $\chi_A(M)$ to be the set of field characteristics over which M is algebraic (i.e., M is algebraic over precisely the characteristics in $\chi_A(M)$).

This definition is motivated by the corresponding linear characteristic set, $\chi_L(M)$, and the study it has received. A summary of some important results about linear sets follows:

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(1) If $0 \in \chi_L(M)$, then $\chi_L(M)$ is cofinite (Rado [10]).

(2) If $\chi_L(M)$ is infinite, then $0 \in \chi_L(M)$ (Vamos [12]).

(3) Every cofinite linear characteristic set (necessarily including 0) is realizable (Reid [11]).

(4) All finite linear characteristic sets (necessarily excluding 0) are realizable (Kahn [6]).

Much less is known about algebraic characteristic sets; we list some results here:

(a) For all matroids M, $\chi_L(M) \subseteq \chi_A(M)$.

(b) If $0 \in \chi_A(M)$, then $0 \in \chi_L(M)$.

(c) The following algebraic characteristic sets are possible:

(i) $\chi_A(M) = \varnothing$ (Ingleton and Main [5]; $M =$ Vamos cube).

(ii) For any prime p, $\chi_A(M) = \{p\}$ (Lindström [8]; $M = L_p$ (the Lazerson matroids)).

(iii) $\chi_A(M) = \{2, 3, 5, \ldots\}$ (everything except 0) (Lindström [9]; $M =$ non-Pappus matroid). (Thus (2) is false for $\chi_A(M)$.)

Both (a) and (b) are long-standing algebraic facts. Note that (a) and (b) together imply (1) above holds for algebraic sets. In Section 3, we show that many non-singleton finite algebraic characteristic sets are possible. At the same time, we also determine $\chi_A(PG(2, p))$ and $\chi_A(AG(2, p))$. In Section 4, we create many new examples of rank 3 non-algebraic matroids and give a result on excluded minors.

The proof of Theorem 2 is modelled after Lindström [8], which reduces an algebraic question to a linear one by using derivations. In fact, this is essentially the same proof technique that shows (b). This result is false for characteristic $p \neq 0$, (consider the non-Fano plane, which is algebraic over any field of characteristic 2 but not linear over any such field) but may be true for large classes of linear matroids.

2. Singleton Algebraic Characteristic Sets

We will need the following algebraic definitions.

Definition. Let F be a field and let x be algebraic over F. Then x is *separable over* F if the minimal polynomial x satisfies over F has no multiple roots. We say an extension field E is *separable* over F if each element of E is separable over F. It is a routine exercise to show that x is separable over F iff $f'(x) \neq 0$, where $f(x)$ is the minimal polynomial for x over F and

$f'(x)$ denotes the formal derivative. (Note we only define separability for algebraic extensions.)

DEFINITION. Let $k \leqslant F \leqslant L$ be fields. A map $D: F \to L$ is called a *derivation of F over k with values in L* if the following three conditions hold:

(1) $D(x)=0$ for all $x \in k$.

(2) $D(x+y)=D(x)+D(y)$ for all $x, y \in F$.

(3) $D(xy)=xD(y)+yD(x)$ for all $x, y \in F$.

The set of all derivations of F over k forms a vector space over F, with dimension equal to the transcendence degree of F over k. More information can be found in [7], for example.

In general, a derivation of F over k with values in L cannot be extended to an extension field E of F. For example, if $F = \mathrm{GF}(2)\ (xy, xz, yz)$, where x, y, and z are independent transcendentals over GF(2) and D is the derivation of F over GF(2) determined by $D(xy)=1$, $D(xz)=0$, and $D(yz)=0$, then the reader may verify that D cannot be extended to $E=\mathrm{GF}(2)(x, y, z)$. The problem here is that the extension field E is not a separable extension of F. The relation between separability and derivation extension is given in the next theorem, which is proven in [7].

THEOREM 1. *Let $F \leqslant E$ be fields, with E separable over F. Then every derivation D of F (with values in some field L) has a unique extension to a derivation of E.*

We now define a class of linear matroids M_p for p prime. Let M_p be the column dependence matroid over GF(p) for the matrix N_p,

$$\begin{array}{ccccccccccc} x_1 & x_2 & x_3 & a_0 & b_0 & a_1 & b_1 & a_2 & b_2 \cdots & a_{p-1} & b_{p-1} \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & p-1 & p-1 \end{bmatrix}$$

M_5 is depicted in Fig. 1.

It is well known that $\chi_L(M_p)=\{p\}$. Further, any minor of M_p is representable (linearly) over characteristic zero. Theorem 2 shows that these facts remain true when "linear" is replaced by "algebraic."

If a matroid of rank r is algebraic over a field F, then we may assume M is algebraic in $\overline{F(x_1,..., x_r)}$ (the algebraic closure of $F(x_1,..., x_r)$), where $x_1,..., x_r$ are algebraically independent transcendentals over F. If $\mathrm{char}(F) \neq 0$, then $\overline{F(x_1,..., x_r)}$ is not a separable extension of $F(x_1,..., x_r)$. Hence it is not true that every derivation of $F(x_1,..., x_r)$ over F can be

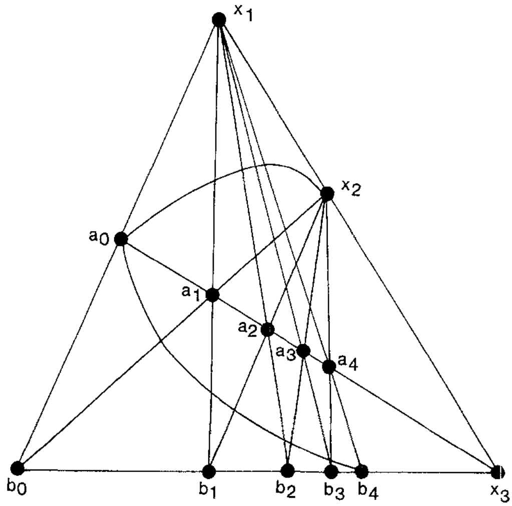

FIGURE 1

extended to $\overline{F(x_1,..., x_r)}$. The proof of the next theorem will involve replacing $\overline{F(x_1,..., x_r)}$ by a smaller field which is separable over $F(x_1,..., x_r)$.

THEOREM 2. $\chi_A(M_p) = \chi_L(M_p) = \{p\}$.

Proof. We know $\chi_L(M_p) \subseteq \chi_A(M_p)$, so we must show containment the other way. Suppose M_p is algebraic over a field F of characteristic q. Then choose an algebraic representation of M_p over F. The rest of the proof will be divided into two parts. First, we show that we may replace the representation selected above by one in which each element is separable over $E = F(x_1, b_0, x_3)$ for some algebraically independent transcendentals x_1, b_0 and x_3. We then use derivations to show $q = p$ and we will be done.

Part 1. Assume that the points of M_p have received algebraic coordinates x_1, x_2, x_3, a_0, b_0, a_1, b_1,..., a_{p-1}, b_{p-1} and this ordering corresponds to the ordering given above. (These elements of $F(x_1, b_0, x_3)$ should not be confused with the labels given to the column vectors in the matrix N_p.) We will replace each of the above coordinates a_i or b_i if necessary by powers of a_i or b_i to obtain a separable representation.

Results of Lindström [8] allow us to assume the first seven points of M_p have been so replaced. (This is just the non-Fano matroid.) We proceed from this point by induction. Assume that all points preceeding a_i ($i \geq 2$) in the ordering given above are separable over E. Define the *degree* of a polynomial f to be the sum over all monomials of all exponents in f. Now $\{a_i, x_2, b_{i-1}\}$ is a circuit, so we choose a polynomial $f \in F[A\ X, B]$ such that $f(a_i^{q^c}, x_2, b_{i-1})=0$ for some integer c, and degree of f minimal. Let $f_j (1 \leqslant j \leqslant 3)$ represent the three formal partial derivatives of f.

Claim. $f_1(a_i^{q^c}, x_3, b_{i-1}) \neq 0$. To see this, suppose the contrary. Then if f_1 were not the zero polynomial, it would have lower degree than f, which is a contradiction. But if f_1 is identically zero, then $f(A, X, B)=g(A^q, X, B)$ for some polynomial g and g would have smaller degree than f, which again is a contradiction. Now replace a_i by $a_i^{q^{c+1}}$. Then a_i is separable over $F(x_1, x_2,..., b_{i-1})$ and hence is separable over E (by induction and the fact that towers of separable extensions are separable). A similar argument works for the b_i and we are done with part 1.

We will need to know that at least one of f_2 or f_3 is nonzero for part 2 of the proof. Suppose $f_j(a_i, x_3, b_{i-1})=0$ for both $j=2$ and 3. Then $f(A, X, B)=g(A, X^{q^d}, B^{q^e})$ for some positive integers d and e. Assume $d \leqslant e$. Now define a new polynomial h from f as follows: Replace all X and B terms by $X^{q^{-d}}$ and $B^{q^{-d}}$, respectively. (For example, if $q=3$, $d=1$, $e=2$, and $f(A, X, B)=A^2+X^3B^9+AX^6B^{18}$, then $h(A, X, B)=A^2+XB^3+AX^2B^6$.) Then h will have smaller degree than f and $[h(a_i^{q^{-d}}, x_2, b_{i-1})]^{q^d}=f(a_i, x_3, b_{i-1})=0$, which contradicts the minimality of f since we could replace a_i by $a_i^{q^{-d}}$. Thus at least one of f_2 or f_3 must be non-zero.

Part 2. Define derivations D_i $(1 \leqslant i \leqslant 3)$ of E over F with values in E by $D_i(y_j)=\delta_{ij}$ (Kronecker delta), where $1 \leqslant j \leqslant 3$ and $y_1=x_1$, $y_2=b_0$, and $y_3=x_3$. If u is separable over E, then define the *gradient vector* Du to be $(D_1(u), D_2(u), D_3(u))$, which is a vector over E. (Note the separability from part 1 is essential here.) Then the $3x(2p+3)$ matrix $N=[D(y_1)^t, D(y_2)^t, D(y_3)^t,..., D(b_{p-1})^t]$ represents a matroid M' linearly over characteristic q (over the field E). It is easy to see M' is a weak map image of M (i.e., any set dependent in M remains dependent in M': If $\{z_1,..., z_k\}$ is dependent in M, then there is a polynomial f with $f(z_1,..., z_k)=0$. Applying D to this equation gives a linear dependence among $\{D(z_1),..., D(z_k)\}$.)

Claim. N is projectively equivalent to the matrix N_p (defined above) over characteristic q. (Two matrices A and B are *projectively equivalent* if $A=NBD$ for some nonsingular matrix N and some nonsingular diagonal matrix D.) To see this, we again proceed by induction. The first seven columns follow from [8]. Now assume the two submatrices of N and N_p determined by points $\{x_1,..., b_{k-1}\}$ are projectively equivalent (for $k>1$). We will show $D(a_k)^t$ is projectively equivalent to $[1, 1, k]^t$.

We now have

$$f(a_k, x_2, b_{k-1})=0, \tag{1}$$

$$g(a_k, a_0, x_3)=0, \tag{2}$$

where f is the polynomial from part 1 and g is another polynomial. Applying D to (1) and (2) gives

$$f_1 D(a_k) + f_2 D(x_2) + f_3 D(b_{k-1}) = [0, 0, 0]^t, \tag{1D}$$

$$g_1 D(a_k) + g_2 D(a_0) + g_3 D(x_3) = [0, 0, 0]^t, \tag{2D}$$

where the partial derivatives are all evaluated at the same points as the original polynomials. The rest of the proof rests on showing almost all of these partial derivatives are nonzero. This will force us to solve linear equations to determine $D(a_k)$—the same equations which were solved in computing the matrix N_p.

Now $f_1 \neq 0$ and at least one of f_2 or $f_3 \neq 0$.

Subclaim. Neither f_2 nor f_3 is zero. If $f_2 = 0$, then $D(a_k)$ is projectively equivalent to $D(b_{k-1})$. We write $D(a_k) = D(b_{k-1})$. Hence $D(x_3)$, $D(a_0)$, and $D(b_{k-1})$ are linearly dependent (since $\{x_3, a_0, a_k\}$ is a circuit). By induction, these three vectors are equivalent to $[1, 1, 0]^t$, $[0, 0, 1]^t$, and $[0, 1, k-1]^t$. But these three vectors are independent over any field and we have a contradiction. But if $f_3 = 0$, then $D(a_k) = D(x_2)$. This forces $D(x_3)$, $D(a_0)$, and $D(x_2)$ to be linearly dependent, which again is impossible. Hence $f_j \neq 0$ for all $1 \leqslant j \leqslant 3$.

Now we can clearly choose g such that at least one of the $g_j \neq 0$ $(1 \leqslant j \leqslant 3)$. But if exactly one of the $g_j \neq 0$, then one of the following holds:

(a) $D(a_0) = [0, 0, 0]^t$,
(b) $D(x_3) = [0, 0, 0]^t$,
(c) $D(a_k) = [0, 0, 0]^t$.

But (a) and (b) are excluded by induction and (c) implies $D(x_2) = D(b_{k-1})$ (from (1D)), which again is excluded by induction. Therefore at least two of the g_j are nonzero. We examine the three possibilities.

(i) $g_1 = 0$. Then, from (2D), $D(a_0) = D(x_3)$. This is impossible by induction.

(ii) $g_2 = 0$. Again, (2D) implies $D(a_k) = D(x_3)$. This forces $D(x_2)$, $D(x_3)$, and $D(b_{k-1})$ to be dependent, which is impossible.

(iii) $g_3 = 0$. Finally, (2D) implies $D(a_k) = D(a_0)$, which in turn gives $D(a_0)$, $D(b_{k-1})$, and $D(x_2)$ dependent, i.e.,

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & k-1 & 1 \end{vmatrix} = k = 0, \quad \text{which occurs iff } q \mid k.$$

In all cases, the vector $[1, 1, k]^t = D(a_k)^t$ and the two matrices remain projectively equivalent over characteristic q. The argument for b_k is similar and we omit it.

Hence, the matroid M' is represented linearly over characteristic q by the matrix N_p considered over characteristic q. Now $\{x_2, a_0, b_{p-1}\}$ is a circuit in M, hence is dependent in M'. But the corresponding three columns in N_p have determinant equal to p. Therefore $q \mid p$, so $q = p$ and we are done.

COROLLARY 3. $\chi_A(PG(2, p)) = \{p\}$.

Proof. This follows from the facts that M_p is a subgeometry of $PG(2, p)$ and $PG(2, p)$ is algebraic over characteristic p.

3. FINITE NON-SINGLETON ALGEBRAIC CHARACTERISTIC SETS

We can repeat the proof of Theorem 2 for the following class of matroids. Let $n = p_1 \cdots p_k + 1$ for given primes $p_1,\ldots, p_k$ and let $s = [\log_2 n]$. For $i = 0, 1, 2,\ldots, s$ set $b_i(n) = b_i = [n/2^{(s-i+1)}]$. Thus $b_0 = 0$, $b_1 = 1$, $b_2 = 2$ or 3, and in general, $b_i = 2b_{i-1}$ or $b_i = 2b_{i-1} + 1$. Note b_i is the integer given by the first i digits in the binary expansion of n. Let $N(n)$ be the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 \cdots 2 & 1 \cdots 2 & & & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & b_i & b_i & b_s & b_s \end{bmatrix}.$$

(This is the general binary construction of [1].) Let $M(n)$ be the column dependence (i.e., linear) matroid of $N(n)$ where dependences are taken over the prime p_1.

THEOREM 4 (Brylawski [1]). $\chi_L(M(n)) \subseteq \{p_1,\ldots, p_k\}$.

This theorem is proven in [1]. We remark that the subdeterminant

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & b_s \end{vmatrix} = 2b_s - 1 = n - 1 = p_1 \cdots p_k.$$

Hence these three columns are dependent over characteristic p_1 (and so in $M(n)$). Therefore $p \in \chi_L(M(n))$ implies $p = p_i$ for some i.

In general, the inclusion is proper. Under certain conditions, however, we get $\chi_L(M(n)) = \{p_1,\ldots, p_k\}$.

THEOREM 5 (Brylawski [1]). *Suppose all the residues b_0, $b_1, ..., b_s$ all differ by at least two modulo each prime p_i (except for b_0 and b_1; perhaps b_1 and b_2). Then $\chi_L(M(n)) = \{p_1, ..., p_k\}$.*

The hypotheses in the above theorem guarantee the zero subdeterminants in $N(n)$ are precisely the same over each prime p_i. We now apply the proof of Theorem 2 to the above class of matroids.

THEOREM 6. *Let $n = p_1 \cdots p_k + 1$ and $N(n)$ and $M(n)$ be defined as above. Further suppose the residues b_0, $b_1, ..., b_s$ satisfy the hypotheses of Theorem 5. Then $\chi_L(M(n)) = \chi_A(M(n)) = \{p_1, ..., p_k\}$.*

The proof is essentially the same as the proof of Theorem 2. We first show that if $M(n)$ is algebraic over characteristic q, then there is a separable algebraic representation. We then apply derivations as before, and get $p_1 p_2 \cdots p_k = 0$ over characteristic q. Hence q divides $p_1 \cdots p_k$ and the proof is complete. We leave the details to the reader.

Note: Theorem 4 is also true when "linear" is replaced by "algebraic."

COROLLARY 7. *$\chi_A(AG(2, p)) = \{0, 2, 3, 5, ...\}$ if $p = 2$ or 3 and $\chi_A(AG(2, p)) = \{p\}$ for $p > 3$.*

Proof. This follows immediately from Proposition 3.5 of [1], which is the analogous result for linear characteristic sets. For $p = 2$ or 3, the result follows from the same fact for $\chi_L(AG(2, p))$. For $p > 5$, note that the matroid $M(p)$ from Theorem 4 is affine since the line $x + y + z = 0$ misses $M(p)$.

EXAMPLE 8. Non-singleton finite algebraic characteristic sets: The computer search used in [1] to find (prime-field linear) characteristic sets is applicable whenever the associated matrix has a subdeterminant equal to the product of the given primes. We list some new algebraic characteristic sets.

(1) Prime pairs: $\{13, 19\}$, $\{23, 59\}$, $\{29, 59\}$, $\{29, 79\}$, $\{29, 157\}$, and many others for $31 \leqslant p, q \leqslant 293$.

(2) Prime triples: $\{71, 193, 797\}$, $\{1009, 1013, 1031\}$, $\{233, 1103, 2089\}$.

(3) Larger sets: The 17 largest primes less than 100,000 form an algebraic characteristic set.

All these examples follow from the methods outlined above or slight modifications of it. In each case, the algebraic and linear characteristic sets coincide.

4. Non-Algebraic Matroids and Minors

We can now construct infinitely many rank 3 non-algebraic matroids. We need the following definition. Suppose M_1 and M_2 are rank 3 matroids. Let M_{12} be the rank 3 matroid $M_{12} = T^3(M_1 \oplus M_2)$, where T represents matroid truncation and $M_1 \oplus M_2$ is the direct sum of M_1 and M_2. (M_{12} is just the matroid obtained by positioning M_1 and M_2 freely in the plane.)

Lemma 9. $\chi_A(M_1) \cap \chi_A(M_2) = \chi_A(M_{12})$.

Proof. If $p \in \chi_A(M_{12})$ then $p \in \chi_A(M_i)$ $(i = 1, 2)$ since M_i is a restriction (deletion minor) of M_{12}. Conversely, since $\chi_A(M_1) \cap \chi_A(M_2) = \chi_A(M_1 \oplus M_2)$ (easy fact) and the truncation of an algebraic matroid is algebraic [13], we have $\chi_A(M_1) \cap \chi_A(M_2) = \chi_A(M_1 \oplus M_2) \subseteq \chi_A(T^3(M_1 \oplus M_2) = \chi_A(M_{12})$.

Corollary 10. *Let $M_1 = M_p$, $M_2 = M_q$, p, q primes, be the matroids defined in Theorem 2. Then $\chi_A(M_{12}) = \varnothing$.*

This gives an infinite family of rank three non-algebraic matroids. Further, each such matroid is minimal; i.e., any minor of M_{12} will be algebraic over (at least) either characteristic p or q. Hence $\{M_{12}\}$ forms an infinite antichain (under minor ordering) of rank three non-algebraic matroids.

Remark 11. It is possible for $\varnothing \neq \chi_L(M)$ finite with $\chi_L(M) \neq \chi_A(M)$. To see this, let M_1 be a matroid with $\chi_L(M_1) = \chi_A(M_1) = \{p_1, p_2, \ldots, p_n\}$, where $p_1 < p_2 < \cdots < p_n$. (See Example 8 from Section 3.) Now let M_2 be the matroid obtained by taking dependences of the matrix N_{p_2} (from Section 2) over the rationals. Then $\chi_L(M_2) = \{0\} \cup \{q : q > p_2\}$ and $\chi_A(M_2) = \{0\} \cup \{\text{all primes}\}$. Let $M = M_1 \oplus M_2$. Then $\chi_L(M) = \{p_2, \ldots, p_n\}$ and $\chi_A(M) = \{p_1, p_2, \ldots, p_n\}$.

Recall a matroid M is an excluded or forbidden minor for representability over a field F if M is not representable over F but any minor of M is. The following proposition addresses excluded minors.

Proposition 12. *There are infinitely many rank three excluded minors for algebraic representability over Q.*

Proof. The family $\{M_p\}$ (as in Theorem 2) gives an infinite collection of matroids, none of which is algebraic over Q. These matroids form an antichain, but any minor of M_p is representable *linearly* over characteristic 0, hence is algebraic over Q (see [9]).

Proposition 12 is related to the following proposition.

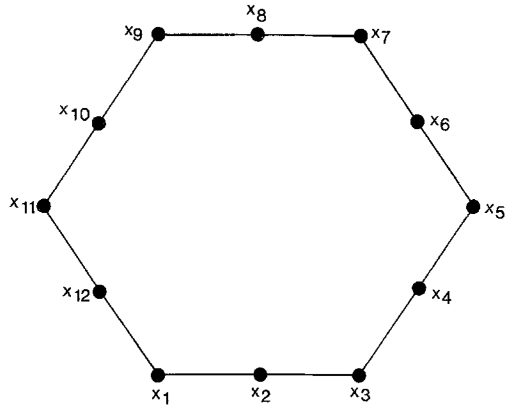

FIGURE 2

PROPOSITION 13. *Let F be any field. Then there is an infinite antichain of matroids, all algebraic over F.*

Proof. Define $G(n)$ on $\{x_1,..., x_{2n}\}$ to be the rank 3 matroid whose 3-element circuits are $\{x_{2i-1}, x_{2i}, x_{2i+1}\} - 1 \leqslant i \leqslant n$, where subscripts are computed modulo $2n$ ($G(6)$ is pictured in Fig. 2). Brylawski shows [2] that $\{G(n)\}$ forms an infinite antichain all linearly representable over Q. Hence $G(n)$ is algebraic over F (see [13]) for all n and we are done.

Propositions 12 and 13 contrast sharply with the corresponding questions concerning linear matroids, both of which are open.

We conclude with some interesting questions concerning algebraic characteristic sets.

(1) If $\chi_L(M) \neq \varnothing$ is finite, prove $\chi_A(M)$ is finite. Methods used in this paper can be extended to prove this under certain conditions.

(2) Are there infinite algebraic characteristic sets which are not cofinite?

(3) What cofinite sets are possible? The only ones presently known are $\{0, 2, 3, 5,...\}$ and $\{2, 3, 5, 7,...\}$ (everything and everything except zero).

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