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# Matroid Automorphisms and Symmetry Groups

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For a subgroup $W$ of the hyperoctahedral group $O_n$ which is generated by reflections, we consider the linear dependence matroid $M_W$ on the column vectors corresponding to the reflections in $W$. We determine all possible automorphism groups of $M_W$ and determine when $W \cong \mathrm{Aut}(M_W)$. This allows us to connect combinatorial and geometric symmetry. Applications to zonotopes are also considered. Signed graphs are used as a tool for constructing the automorphisms.

## 1. Introduction

Combinatorics is a powerful tool which has been applied to other branches of mathematics, frequently exposing some fundamental structure. In this paper, we are concerned with the connections between combinatorial and geometric symmetry, where groups are used to model both kinds of symmetry. In particular, we will interpret various symmetry groups combinatorially, via graphs, signed graphs and matroids. We then compare the geometric symmetry group of an $n$-dimensional polytope (frequently a hypercube or a simplex) to the automorphism group of the associated combinatorial structure – this is our interpretation of combinatorial symmetry. In all of the cases we will consider, there will be close ties between these two groups. In particular, these two groups will coincide for two important classes.

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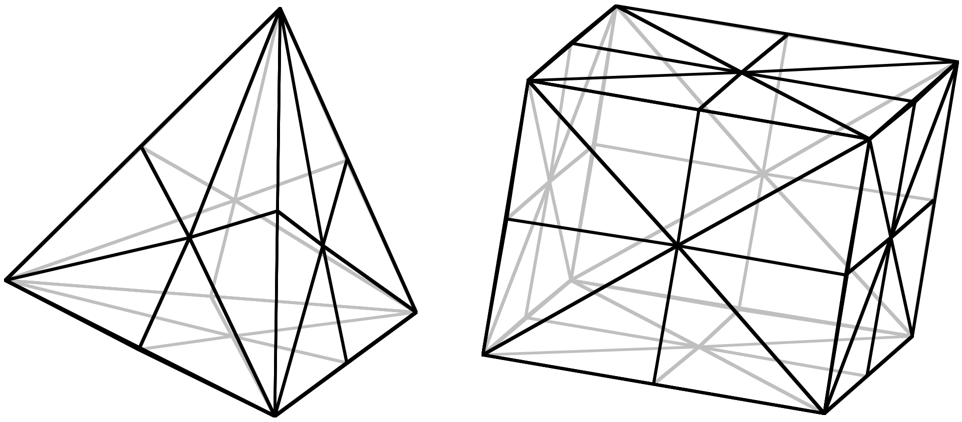

*Figure 1*

Our motivation throughout this paper is the following question – how much of the geometric symmetry of an $n$-dimensional polytope is encoded in the purely combinatorial linear dependence information of the hyperplanes of symmetry? In general, this dependence structure gives a matroid. When can we uniquely recover the symmetry group from this matroid?

Thus, we will be concerned with the application of combinatorial techniques to two reflection (or Coxeter) groups: the symmetry groups of the $n$-simplex and the $n$-dimensional hypercube. These groups are extremely well studied and of fundamental importance in many areas. The symmetry group of the $(n-1)$-simplex is just $S_n$, the symmetric group. The symmetry group of the $n$-dimensional hypercube, the *hyperoctahedral* group, is isomorphic to a semi-direct product $\mathbf{Z}_2^n \rtimes S_n$. See Figure 1 for a picture of the tetrahedron and the cube, together with all their planes of symmetry.

The hyperoctahedral group can be viewed in many different ways. The associated real hyperplane arrangement determines $2^n n!$ maximum dimension regions [1, 9]. This result has interpretations involving Lie algebras (via root systems), non-Radon partitions of finite point sets (the number of ways to partition a point set using one hyperplane), zonotopes (the number of vertices), oriented matroids, signed graphs, symmetry groups (the order of the hyperoctahedral group), and the characteristic polynomial of an associated matroid (via the standard deletion/contraction algorithm). See [1, 6, 7, 10] for detailed accounts of these applications.

Our technique for analysing these two symmetry groups concentrates on the reflections: these generate the symmetry groups, but they also possess a meaningful combinatorial structure. In particular, with each reflection $R$ in a given symmetry group $W$ we will associate a normal vector $\mathbf{c_R}$ of the reflecting hyperplane. When we arrange these normal vectors as column vectors of a matrix $N_W$, we obtain a column dependence matroid $M_W$. When $W$ is a subgroup of the symmetry group of the $n$-simplex, $M_W$ is isomorphic to the cycle matroid of an ordinary graph. When $W$ is a subgroup of the symmetry group of the $n$-dimensional hypercube, $M_W$ is isomorphic to the matroid associated with a signed graph. In fact, the matroid associated with the hypercube is a Dowling geometry. These graphs, which we call $\Gamma_W$, will help us determine the automorphism groups of the associated matroids. See [2] for more information on Dowling geometries and their automorphisms.

Thus, we are concerned with the following objects and their various interrelations:

(a) an $n$-dimensional polytope (usually either the simplex $s_{n-1}$ or the hypercube $Q_n$);
(b) a subgroup $W$ of the symmetry group of the polytope;
(c) a matrix $N_W$ of normal vectors of the reflecting hyperplanes of $W$;
(d) a graphic object $\Gamma_W$;
(e) a matroid $M_W$.

Our main interest is the connection between $W$ and $\mathrm{Aut}(M_W)$ or $\mathrm{Aut}(\Gamma_W)$, as we study the (matroidal) dependence structure of the normal vectors of the reflecting hyperplanes. Theorem 4.7 shows that this dependence structure is sufficient to determine the geometric symmetry of the polytope, that is, all matroid automorphisms of $M_W$ 'correspond' to symmetries of $W$.

Section 2 is concerned with the simplex. Although this treatment of the simplex is properly a subcase of our treatment of the hypercube, we include it here because it is independent of signed graph and matroid theory, depending only on ordinary graph theory. Theorem 2.3 describes the graphic automorphism groups that can arise from reflection subgroups of $S_n$, the symmetry group of the $(n-1)$-simplex.

Sections 3 and 4 treat the more general case of reflection subgroups of $O_n$, the hyperoctahedral group. Our main result, Theorem 4.7, develops a crystallographic interpretation of the automorphism groups of these matroids, explaining how the geometric symmetry information is encoded in the associated matroid structure. Signed graphs are used extensively throughout these sections; group closure and matroid automorphisms are computed via signed graphs.

Section 5 gives an interpretation of the main automorphism results in geometric terms. We develop a direct correspondence between the symmetry operations on the hypercube and matroid automorphisms, allowing us to understand the connection between geometric and combinatorial symmetry in a different way.

Section 6 concludes with a more geometric connection involving zonotopes and the Platonic solids. Given a subgroup $W$ of the symmetry group of a regular polytope $P$, we first associate a matrix $N_W$, then a zonotope $Z_W$. Instead of computing a combinatorial automorphism group (as in Sections 2–4), however, we determine the geometric symmetry group $\mathrm{Sym}(Z_W)$. This essentially short-circuits the combinatorics, associating one geometric symmetry group with another. In all cases we consider, there is a close connection between $W$ and $\mathrm{Sym}(Z_W)$.

We assume the reader is familiar with graph theory and the basics of matroid theory. A standard matroid reference is [11].

## 2. The $n$-simplex and graphs

Let $s_{n-1}$ be the $(n-1)$-simplex, realized in $\Re^n$ as the convex hull of $\mathbf{e}_1, \dots, \mathbf{e}_n$, the unit vectors that comprise the standard basis for $\Re^n$. Then $s_{n-1}$ is contained in the hyperplane $\sum x_i = 1$. We number the vertices of the simplex $1, \dots, n$ and note that the hyperplanes representing reflections in the symmetry group of $s_{n-1}$ have equations of the form $x_i = x_j$, for $1 \leqslant i < j \leqslant n$. The reflection corresponding to $x_i = x_j$ has the effect of interchanging the vertices $i$ and $j$; we therefore represent this reflection by the transposition $(ij)$. Then the collection of all $\binom{n}{2}$ such reflections clearly generates the symmetric group $S_n$. See Figure 1 for a picture of the tetrahedron and its reflection planes.

Let $H$ be a subgroup of $S_n$ generated by transpositions: then $H$ is a *reflection* subgroup of $\mathrm{Sym}(s_{n-1})$. There is a natural way to associate a graph $\Gamma_H$ with $H$: the $n$ vertices of $\Gamma_H$ are labelled $1, \ldots, n$ with vertices $i$ and $j$ joined by an edge $e_{ij}$ if and only if the reflection represented by the transposition $(ij)$ is in $H$.

This graph has been studied before, especially from the viewpoint of hyperplane arrangements. See [13, 14, 15, 16, 17] for more details. We are interested in how the graph gives information about the group-theoretic structure of $H$.

**Question 1.** What is the relation between the subgroup $H$ and the automorphism group $\mathrm{Aut}(\Gamma_H)$?

Although the answer to this question follows from our work on the hypercube below, we include a direct treatment here. This allows us to avoid reference to matroids and concentrate on graphs. We begin by showing that group-theoretic closure in $H$ corresponds precisely to cycle closure in $\Gamma_H$.

**Proposition 2.1 (Group closure in $\Gamma_H$).** *Let $H$ be a subgroup of $S_n$ that is generated by transpositions and let $\Gamma_H$ be the graph associated with $H$. If $e_{ij}, e_{jk} \in E(\Gamma_H)$, then $e_{ik} \in E(\Gamma_H)$.*

**Proof.** If both $(ij)$ and $(jk)$ are elements of $H$, then so is the conjugate $(ij)(jk)(ij) = (ik)$. □

Thus each component of $\Gamma_H$ is a clique. Let $\kappa_i$ equal the number of components of $\Gamma_H$ having exactly $i$ vertices. Then an immediate corollary is the following.

**Corollary 2.2.** *Let $H$ be a subgroup of $S_n$ that is generated by transpositions. Then*

$$H \cong \prod_{i=1}^{n} S_i^{\kappa_i}$$

*where $\sum i\kappa_i = n$.* □

This is very well known. We can now give a complete description of $\mathrm{Aut}(\Gamma_H)$. This result is also well known. We justify its inclusion here to contrast our straightforward graphic approach with the more involved matroidal approach of Sections 3 and 4.

**Theorem 2.3.** *Let $H$ be a subgroup of $S_n$ that is generated by transpositions and let $\Gamma_H$ be the graph associated with $H$, where $\Gamma_H$ has $\kappa_i$ components on $i$ vertices. Then*

$$\mathrm{Aut}(\Gamma_H) \cong \prod_{i=1}^{n} \left[ S_i^{\kappa_i} \rtimes S_{\kappa_i} \right].$$

*In particular, $H \lhd \mathrm{Aut}(\Gamma_H)$ and $[\mathrm{Aut}(\Gamma_H) : H] = \prod_{i=1}^{n} \kappa_i!$.*

**Proof.** Since $\Gamma_H$ is a disjoint union of cliques, we clearly have $H \cong \prod_{i=1}^{n} S_i^{\kappa_i} \leqslant \mathrm{Aut}(\Gamma_H)$. A graph automorphism can also permute any collection of components of the same size: thus $S_{\kappa_i} \leqslant \mathrm{Aut}(\Gamma_H)$. Finally, an element $\left(\sigma_1, \ldots, \sigma_{\kappa_i}\right) \in S_i^{\kappa_i}$ acts in $\mathrm{Aut}(\Gamma_H)$ by permuting each of the $\kappa_i$ $i$-cliques of $\Gamma_H$ individually. It is straightforward to check that conjugation of this element by $\pi \in S_{\kappa_i}$ has the effect of applying the permutation $\pi$ to the $\kappa_i$ cliques, that is, $\pi^{-1}\left(\sigma_1, \ldots, \sigma_{\kappa_i}\right)\pi = \left(\sigma_{\pi(1)}, \ldots, \sigma_{\pi(\kappa_i)}\right)$. This is precisely the group-theoretic property we need to get a semi-direct product. This also shows $H \triangleleft \mathrm{Aut}(\Gamma_H)$. □

Note that isolated points of the graph contribute to $\mathrm{Aut}(\Gamma_H)$. A matroidal interpretation would ignore isolated points, since the matroid is defined on the edge set. We explore the connections with matroids in Section 4. See also the comments following Theorem 4.7.

## 3. Hypercubes, groups and signed graphs

Let $Q_n$ be the $n$-dimensional hypercube with vertices $(\pm 1, \ldots, \pm 1)$. Then we denote the symmetry group of $Q_n$, the *hyperoctahedral group*, by $O_n$. This symmetry group contains $n^2$ reflections, where each reflection is determined by a hyperplane whose equation has the form $x_i = 0$ or $x_i = \pm x_j$, for $1 \leqslant i, j \leqslant n$. See Figure 1 for a picture of these hyperplanes and the cube in $\mathfrak{R}^3$.

There are many ways to represent the reflections in $O_n$. Let $i$ represent the positive $x_i$ axis and $i^*$ represent the negative $x_i$ axis. Then we can realize $O_n$ as a permutation group on the symbols $\{1, 1^*, 2, 2^*, \ldots, n, n^*\}$ as follows.

(a) The reflection whose equation is given by $x_i = 0$ flips the $x_i$ axis and fixes all other axes. We represent this reflection by the permutation $(ii^*)$.
(b) The reflection with equation $x_i = x_j$ interchanges the $x_i$ and $x_j$ axes and fixes all other axes. We represent this reflection by the permutation $(ij)(i^*j^*)$.
(c) The reflection with equation $x_i = -x_j$ also interchanges the $x_i$ and $x_j$ axes, but in the opposite sense. We represent this reflection by the permutation $(ij^*)(i^*j)$.

We are interested in subgroups $W$ of $O_n$ generated by reflections. Let $\{R_1, \ldots, R_k\}$ be the set of all reflections in a subgroup $W$ with corresponding normal vectors $\{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$. Finally, let $M_W$ be the matroid on these normal vectors, with dependences taken over the rationals. We will be concerned with the following question.

**Question 2.** What is the connection between the subgroup $W$ and $\mathrm{Aut}(M_W)$?

Our approach to this question uses signed graphs. A *signed graph* is a graph (with loops and multiple edges allowed) in which each edge is labelled with a '+' or '−' sign. Signed graphs were introduced by Harary [8] and have been studied extensively by Zaslavsky from a matroid-theoretic viewpoint [13, 14, 15, 16, 17].

Signed graphs will be useful for us because they naturally represent the three kinds of reflection that may be present in a subgroup $W$ of $O_n$: $x_i = 0, x_i = x_j$ and $x_i = -x_j$. In particular, we construct a signed graph $\Gamma_W$ with vertices $V(\Gamma_W) = \{1, 2, \ldots, n\}$, where the vertex $i$ corresponds to the $x_i$-coordinate axis. We join vertices $i$ and $j$ by an edge signed '+' if the reflection $x_i = x_j$ is in $W$ and by an edge signed '–' if the reflection $x_i = -x_j$ is in $W$. We also place a negatively signed loop at the vertex $i$ if the reflection $x_i = 0$ is in $W$. (Since the reflection $x_i = 0$ can be written $x_i = -x_i$, signing all loops negatively is consistent with edge signing.) Denote these three kinds of edges by $e_{ij}^+, e_{ij}^-$ and $e_i^-$, respectively, and the edge set by $E(\Gamma_W)$.

We briefly recall a few definitions about balance in signed graphs [13]. Let $\Gamma$ be a signed graph, possibly including multiple edges and loops. A *balanced circle* is a set of edges forming a cycle $C$ in the underlying unsigned graph such that the product of all the edge-signs in $C$ is positive. A set of edges $S$ is *balanced* if it contains no unbalanced circles.

In order to use signed graphs here, we need to understand group-theoretic closure in this context.

**Proposition 3.1 (Group closure in $\Gamma_W$).** *Let $W$ be a subgroup of $O_n$ that is generated by reflections, and let $\Gamma_W$ be the signed graph associated with $W$.*

(1) *If $e_{ij}^\alpha, e_{jk}^\beta \in E(\Gamma_W)$, then $e_{ik}^{\alpha\beta} \in E(\Gamma_W)$, where $\alpha$ and $\beta$ represent '+' or '–' and the product $\alpha\beta$ is computed in the obvious way.*
(2) *If $e_i^-$ is a loop and $e_{ij}^+$ is an edge of $\Gamma_W$, then $e_{ij}^-$ is in $\Gamma_W$.*
(3) *If $e_i^-$ is a loop and $e_{ij}^-$ is an edge of $\Gamma_W$, then $e_{ij}^+$ is in $\Gamma_W$.*
(4) *If $e_i^-$ is a loop and either $e_{ij}^+$ or $e_{ij}^-$ is an edge of $\Gamma_W$, then the loop $e_j^-$ is in $\Gamma_W$.*

**Proof.** We prove parts (2), (4) and one case of part (1); the proof of the other cases of part (1) are similar to the proof we present, and the proof of (3) is similar to that of (2).

**(1)** Taking $\alpha = \beta = +$ and translating this back to the subgroup $W$, we have $(ij)(i^*j^*)$ and $(jk)(j^*k^*)$ in $W$, so the conjugate

$$(ij)(i^*j^*)(jk)(j^*k^*)(ij)(i^*j^*) = (ik)(i^*k^*) \in W.$$

**(2)** If $e_{ij}^+$ is an edge of $\Gamma_W$, then we have $(ii^*)$ and $(ij)(i^*j^*)$ in $W$, so the conjugate

$$(ii^*)(ij)(i^*j^*)(ii^*) = (ij^*)(i^*j) \in W.$$

**(4)** If $e_{ij}^+ \in E(\Gamma_W)$, we have $(ii^*)$ and $(ij)(i^*j^*)$ in $W$, so the conjugate

$$(ij)(i^*j^*)(ii^*)(ij)(i^*j^*) = (jj^*) \in W.$$

The argument is similar if $e_{ij}^-$ is an edge of $\Gamma_W$. □

There are three possible connected signed graphs which can occur as $\Gamma_W$ for a reflection subgroup $W$ of $O_n$.

**Definition 1.** Let $r \geqslant 1$.

***$K_r(A, B)$* ('Slim').** This is a balanced signed complete graph. Ignoring signs, this graph is isomorphic to the complete graph $K_r$. The signs of the edges are determined by partitioning the $r$ vertices into two subsets, $A$ and $B$, where $e$ is negatively signed if and only if $e$ joins a vertex in $A$ to a vertex in $B$.

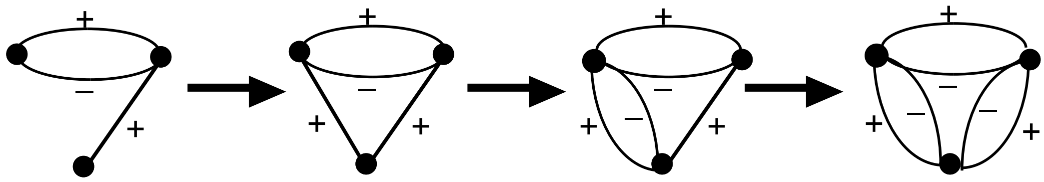


*Figure 2*

**$\pm K_r$ ('Puff').** In this graph, every pair of vertices $i, j$ is joined by both $e_{ij}^{+}$ and $e_{ij}^{-}$.
**$\pm K_r^{\circ}$ ('Superpuff').** Same as $\pm K_r$ with loops adjoined to every vertex.

Note that the edge set of the balanced signed complete graph $K_r(A, B)$ is balanced, that is, every cycle is positive. In fact, this property is equivalent to our definition of $K_r(A, B)$ by a straightforward argument.

The next result follows from repeated application of Proposition 3.1.

**Proposition 3.2.** *Let $W \leqslant O_n$ be generated by reflections. Then each connected component of $\Gamma_W$ is isomorphic to $K_r(A, B), \pm K_r$, or $\pm K_r^{\circ}$, where $1 \leqslant r \leqslant n$.*

**Sketch of proof.** ***$K_r(A, B)$*.** If a component of $\Gamma_W$ contains no multiple edges or loops, then it can be filled out to a signed complete graph by using Proposition 3.1(1). It is straightforward to check the signs obey the partitioning condition.

**$\pm K_r$.** If a component of $\Gamma_W$ contains multiple edges (but no loops), then Proposition 3.1(1) will force this component to contain $\pm K_r$. See Figure 2 for an example when $r = 3$.

**$\pm K_r^{\circ}$.** If a component contains loops, then parts (2) and (3) of Proposition 3.1 will double all edges, while part (4) will place loops at all vertices of the component. □

Thus, there are only three irreducible subgroups which can appear as reflection subgroups of $O_n$. This is closely related to the classification of irreducible root systems: see [9], for example. We list these subgroups in the next proposition.

**Proposition 3.3.**

(1) *If $\Gamma_W = K_r(A, B)$, then $W \cong S_r$, the symmetric group.*
(2) *If $\Gamma_W = \pm K_r$, then $W \cong \mathbf{Z}_2^{r-1} \rtimes S_r$, the Coxeter group associated with the irreducible root system $D_r$. We call this group $O_r^{+}$.*
(3) *If $\Gamma_W = \pm K_r^{\circ}$, then $W \cong \mathbf{Z}_2^{r} \rtimes S_r$, the Coxeter group associated with the irreducible root system $B_r$, that is, the symmetry group $O_r$ of an $r$-dimensional hypercube.*

Note that $O_r^{+}$ is an index 2 subgroup of $O_r$. $O_r^{+}$ is obtained from $O_r$ by removing the reflections whose reflecting hyperplanes have equations $x_i = 0$, that is, the coordinate hyperplanes, from the reflections of $O_r$. In Section 5 we elaborate on how the semi-direct product structure is visible geometrically. See [4] for more information on these symmetry groups.

## 4. Matroid automorphisms and signed graphs

We are now ready to consider the matroid structure on the signed graph $\Gamma_W$ obtained from the subgroup $W$. Recall that $M_W$ is the linear dependence matroid whose elements correspond to the normal vectors of the reflecting hyperplanes. For example, $M_{O_4}$ is the matroid whose elements are the column vectors of the matrix

$$N_{O_4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

Note that the choice of coordinates for the normal vectors has no effect on the matroid; for consistency, we will always choose normal vectors having at most 2 nonzero entries, where each nonzero entry is 1 or $-1$. Allowing the rows of the matrix to correspond to vertices and the columns to edges, the matrix gives the vertex-edge incidence structure for the signed graph $\Gamma_W$. It is obvious that every signed graph has such a matrix representation. Thus, every signed graph $\Gamma$ has an associated linear dependence matroid $M_\Gamma$.

We will determine the automorphism groups $\mathrm{Aut}(M_W)$ for each of the three reflection subgroups $W$ for which $\Gamma_W$ is a connected signed graph. To see the graphical structure of this matroid, we state the following well-known characterization [13].

**Proposition 4.1.** *Let $\Gamma$ be a signed graph with edge set $E$, and let $C \subseteq E$. Then $C$ is a circuit in the matroid $M_\Gamma$ if and only if*

(1) *$C$ is a balanced circle, or*
(2) *$C$ is the union of two edge-disjoint unbalanced circles together with a simple connecting path (of length $k \geqslant 0$) meeting each unbalanced circle in precisely one vertex.*

We could equally well take the characterization of Proposition 4.1 as our definition of the matroid $M_W$, and then prove this matroid coincides with the vector matroid of the associated matrix. We will use both characterizations of $M_W$ below.

The next result shows that the matroid associated with the balanced signed complete graph is equivalent to the cycle matroid of an unsigned complete graph. The proof follows immediately from Proposition 4.1, since the edge set of $K_n(A, B)$ is balanced.

**Lemma 4.2.** *Let $W \leqslant O_n$ with $\Gamma_W = K_n(A, B)$, the balanced signed complete graph. Then $M_W \cong M(K_n)$, the cycle matroid on the complete graph. Thus, the matroid structure on $K_n(A, B)$ is independent of the partition $(A, B)$.* □

The next result shows that, for $\Gamma_W = \pm K_n$, every matroid automorphism of $M_W$ is a signed graph automorphism of $\Gamma_W$. Thus, we can concentrate solely on signed graph automorphisms when determining $\mathrm{Aut}(M_W)$.

**Lemma 4.3 (Matroid automorphisms preserve Puff).** *Let $\Gamma_W = \pm K_n$ and let $e^+, e^-$ be a pair of parallel edges in $\Gamma_W$. If $\varphi : M_W \to M_W$ is a matroid automorphism, then $\varphi(e^+)$*

*and $\varphi(e^-)$ correspond to parallel edges in $\Gamma_W$. Further, if $e$ and $f$ are adjacent, non-parallel edges, then $\varphi(e)$ and $\varphi(f)$ are also adjacent, non-parallel edges.*

**Proof.** First note that if $n = 2$ there are no non-parallel edges, so the result is trivial. For $n > 2$ (and ignoring signs), there are three possibilities for the edges $\varphi(e^+)$ and $\varphi(e^-)$ in $\Gamma_W$ (where $e^+, e^-$ are a pair of parallel edges): parallel edges, adjacent but not parallel edges, or non-adjacent edges (for $n > 3$). By considering the size of various circuits in $M_W$, we will show these last two cases can not occur.

To see why $\varphi(e^+)$ and $\varphi(e^-)$ cannot be adjacent, non-parallel edges, we note that (by Proposition 4.1) there are no three-point circuits in $M_W$ containing both $e^+$ and $e^-$, but there is exactly one three-point circuit containing a pair of adjacent, non-parallel edges. Thus $\varphi(e^+)$ and $\varphi(e^-)$ do not correspond to adjacent, non-parallel edges.

To see why $\varphi(e^+)$ and $\varphi(e^-)$ cannot correspond to non-adjacent edges, note that (by Proposition 4.1) there are exactly $2n-4$ four-point circuits in $M_W$ containing both $e^+$ and $e^-$, but there are exactly 2 four-point circuits containing a pair of non-adjacent edges. Thus, unless $2n-4=2$, these two pairs of edges cannot correspond under a matroid automorphism. But $2n-4=2$ implies $n=3$, in which case there are no non-adjacent edges.

Thus, the pair $\varphi(e^+)$ and $\varphi(e^-)$ correspond to a pair of parallel, oppositely signed edges in $\Gamma_W$.

For the second incidence relation, note that 2 pairs of incident, parallel edges form a four-point circuit in $M_W$, but pairs of parallel edges which share no vertices form an independent set. Since pairs of parallel edges are mapped to pairs of parallel edges by the above proof (and $\varphi$ preserves matroid dependence), pairs of incident, parallel edges remain coincident. □

The next lemma shows that matroid automorphisms preserve the graphic structure on $W = \pm K_n^\circ$, too. The matroid $M_W$ is a Dowling geometry in this case. See [2] for more details on automorphisms of Dowling geometries.

**Lemma 4.4 (Matroid automorphisms preserve Superpuff).** *Let $W = \pm K_n^\circ$ with $n > 2$ and let $f$ be a loop which is adjacent to the pair of parallel edges $e^+, e^-$ in $\Gamma_W$. If $\varphi : M_W \to M_W$ is a matroid automorphism, then*

(1) *$\varphi(f)$ corresponds to a loop in $\Gamma_W$;*
(2) *$\varphi(e^+)$ and $\varphi(e^-)$ correspond to parallel edges in $\Gamma_W$;*
(3) *$\varphi(f)$ is adjacent to the parallel pair $\varphi(e^+), \varphi(e^-)$.*

**Proof. (1)** As in the proof of Lemma 4.3, we examine the number of circuits of various sizes in $M_W$. Note that there are two distinct ways three-point circuits containing the given loop $f$ can be formed:

- $f$ together with two adjacent, parallel edges;
- $f$ with one edge and a loop adjacent to the other endpoint of this edge.

There are $n-1$ circuits of the first type and $2n-2$ of the second type. Thus, there are $3n-3$ three-point circuits containing the loop $f$.

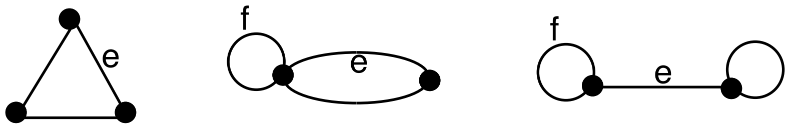


*Figure 3*

For a given non-loop $e$, there are three distinct ways to form a three-point circuit containing $e$:

- $e$ together with two non-loop edges which form a triangle in which there are an even number of negatively signed edges;
- $e$ with its parallel edge and a loop adjacent to an endpoint of $e$;
- $e$ with two loops adjacent to its endpoints.

By Proposition 4.1, there are $2n - 4$ circuits of the first type, 2 of the second type and 1 of the third. This yields a total of $2n - 1$ three-point circuits containing the edge $e$. See Figure 3 for a picture of the three types of three-point circuits of $M_W$.

Thus, $\varphi$ maps loops to loops provided $3n - 3 \neq 2n - 1$, that is, $n \neq 2$. (This exception is not a deficiency in our proof: $n = 2$ is an exceptional case. See Theorem 4.6.)

**(2)** The proof is similar to the proof of Lemma 4.3 and we omit it.

**(3)** Note that $\{e^+, e^-, f\}$ is a circuit in $M_W$, so $\{\varphi(e^+), \varphi(e^-), \varphi(f)\}$ is also a circuit. But the only three-point circuits using 1 loop and 2 parallel edges have the loop adjacent to the parallel edges. □

We are now ready to give a complete interpretation of matroid automorphisms via signed graphs.

**Proposition 4.5.** *Let $W$ be a subgroup of $O_n$ ($n > 2$) that is generated by reflections, with $\Gamma_W$ a connected signed graph.*

(1) *If $\Gamma_W = K_n(A, B)$, then every automorphism of $M_W$ corresponds to a permutation of the vertices of $K_n(A, B)$.*

(2) *If $\Gamma_W = \pm K_n$, then every automorphism of $M_W$ corresponds to a permutation of the vertices of $\pm K_n$, followed by an exchanging ($e^+ \leftrightarrow e^-$) of each pair of parallel edges forming a cutset of $\pm K_n$.*

(3) *If $\Gamma_W = \pm K_n^\circ$, then every automorphism of $M_W$ corresponds to a permutation of the vertices of $\pm K_n^\circ$, followed by an exchanging ($e^+ \leftrightarrow e^-$) of each pair of parallel edges forming a cutset of $\pm K_n^\circ$.*

**Proof.** **(1)** By Lemma 4.2, the matroid $M_W$ is the cycle matroid on $K_r$. Then $K_r$ is 2-connected (since $r > 2$), so Whitney's theorem (Sec. 5.3 of [11]) ensures that matroid automorphisms correspond to graph automorphisms. (It is also easy to show this directly by associating minimal bonds of $M_W$ with vertices of $K_r$.)

**(2)** We first show that vertex permutations and edge exchanges in cutsets are matroid automorphisms. Recall that $N_W$ is the $(0, 1, -1)$ matrix of order $n \times (n^2 - n)$ representing $M_W$

and let $P$ be an $n \times n$ permutation matrix. Then the matrix product $PN_W$ still represents $M_W$, and multiplication by $P$ has the effect of permuting those rows of $N_W$ corresponding to the vertices of $\Gamma_W$. This row permutation induces a permutation of the columns in precisely the same way that a vertex permutation in $\Gamma_W$ induces an edge permutation.

To show that exchanging each pair of edges ($e^+ \leftrightarrow e^-$) in a cutset is a matroid automorphism, we note that this operation preserves all the circuits of the matroid, by Proposition 4.1. If $C$ is a balanced circle, then $C$ will remain balanced after an edge exchange since an even number of exchanges will occur within $C$. If $C$ is a disjoint union of two unbalanced circles $C_1$ and $C_2$ joined by a simple path $P$, then $C$ will retain this structure after the edge exchanges because an even number of exchanges will occur within $C_1$ and within $C_2$. (The exchanges within $P$ are irrelevant.)

We must now show that every matroid automorphism is obtained in this manner. By Lemma 4.3, we know that matroid automorphisms permute vertices and exchange pairs of parallel edges $e^+ \leftrightarrow e^-$. We must show that the collection of edges which are exchanged forms a cutset in $\Gamma_W$. Suppose $e^+ \leftrightarrow e^-$ is a pair of parallel edges which are exchanged by a matroid automorphism $\varphi$ and let $u, v$ be the two endpoints of the pair $e^+, e^-$. Further assume that the vertex permutation is trivial. If there is a path $f_1^+, \ldots, f_k^+$ from $u$ to $v$ in $\Gamma_W$ consisting of edges which are not exchanged by $\varphi$, then the matroid circuit $\{e^+, f_1^+, \ldots, f_k^+\}$ is mapped by $\varphi$ to $\{e^-, f_1^+, \ldots, f_k^+\}$, which is not a matroid circuit. Thus, there is no path from $u$ to $v$ consisting solely of edges that are not exchanged by $\varphi$, so the collection of exchanged edges forms a cutset.

**(3)** This proof proceeds precisely as (2), since the loop–parallel edge incidence relation is preserved by matroid automorphisms (by Lemma 4.4). □

We are now ready to prove the main results of this section. We treat the case where $\Gamma_W$ is connected; the general case will follow.

**Theorem 4.6.** *Let $W$ be a subgroup of $O_n$ that is generated by reflections such that $\Gamma_W$ is a connected signed graph. For $n > 2$:*

(1) *if $\Gamma_W = K_n(A,B)$, then* $\mathrm{Aut}(M_W) \cong W \cong S_n$*;*
(2) *if $\Gamma_W = \pm K_n$, then* $\mathrm{Aut}(M_W) \cong W \cong O_n^+$*;*
(3) *if $\Gamma_W = \pm K_n^\circ$, then* $\mathrm{Aut}(M_W) \cong O_n^+$.

*For $n = 2$:*

(1) *if $\Gamma_W = K_2(A,B)$, then* $\mathrm{Aut}(M_W) \cong \{e\}$*;*
(2) *if $\Gamma_W = \pm K_2$, then* $\mathrm{Aut}(M_W) \cong \mathbf{Z}_2$*;*
(3) *if $\Gamma_W = \pm K_2^\circ$, then* $\mathrm{Aut}(M_W) \cong S_4$.

**Proof.** **(1)** By Lemma 4.2 and Proposition 4.5(1), we have $\mathrm{Aut}(M_W) \cong S_n$ when $n > 2$. When $n = 2$, there is only one element in the matroid $M_W$, so the automorphism group is trivial.

**(2)** We know $W \cong \mathbf{Z}_2^{n-1} \rtimes S_n$; but the matroid automorphisms of Proposition 4.5(2) give precisely the same group-theoretic structure. Each cutset edge exchange operation of

Proposition 4.5(2) has order 2, the order of these exchanges is irrelevant and there are precisely $2^{n-1}$ cutsets in $\Gamma_W$. Thus, the subgroup of Aut($M_W$) generated by these cutset exchanges is isomorphic to $\mathbf{Z}_2^{n-1}$, and this subgroup is normal in Aut($M_W$). Further, the subgroup $S_n$ (considered as the vertex permutation group of the signed graph) is a semi-direct factor of Aut($M_W$); if $\psi_C$ exchanges all the parallel edge pairs in a cutset $C$ and $\pi$ is any vertex permutation, then the conjugate $\pi^{-1}\psi_C\pi = \psi_{\pi(C)}$, the matroid automorphism which exchanges the parallel edges in the image cutset $\pi(C)$. Finally, when $n = 2, M_W$ is the two-point matroid of rank two, so $\mathrm{Aut}(M_W) \cong \mathbf{Z}_2$.

**(3)** When $n > 2$, the argument proceeds exactly as in part (2). For $n = 2$, note that $M_W$ is the rank 2 uniform matroid on four points; that is, the four-point line. Thus, $\mathrm{Aut}(M_W) \cong S_4$. □

We conclude the section by completely answering Question 2 when $\Gamma_W$ may be disconnected. The proof, which we omit, uses Theorem 4.6.

**Theorem 4.7.** *Let $W$ be a subgroup of $O_n$ that is generated by reflections and let $\Gamma_W$ be the graph associated with $W$, where*

**Slim:** *$\Gamma_W$ has $a_i$ components of type $K_i(A, B)$;*
**Puff:** *$\Gamma_W$ has $b_i$ components of type $\pm K_i$;*
**Superpuff:** *$\Gamma_W$ has $c_i$ components of type $\pm K_r^\circ$.*
*Then*

$$\begin{aligned}\mathrm{Aut}(M_W) \quad \cong \quad & S_{a_2} \times \left[\mathbf{Z}_2 \rtimes S_{b_2}\right] \times \left[S_4 \rtimes S_{c_2}\right] \times \prod_{i=3}^{n} \left[S_i^{a_i} \rtimes S_{a_i}\right] \times \\ & \prod_{i=3}^{n} \left[\left(\mathbf{Z}_2^{i-1} \rtimes S_i\right) \rtimes S_{b_i}\right] \times \prod_{i=3}^{n} \left[\left(\mathbf{Z}_2^{i-1} \rtimes S_i\right) \rtimes S_{c_i}\right].\end{aligned}$$ □

Note that isolated points of $\Gamma_W$ do not contribute to Aut($M$) because these vertices are not represented in the matroid $M_W$. This is the right way to treat isolated vertices from the group-theoretic viewpoint; if $i$ is an isolated vertex of $\Gamma_W$, then no reflection involving the facets $i$ or $i^*$ of the hypercube is present in the subgroup $W$. This is the only case where the signed graph does not model the matroid adequately. See the comments following Theorem 2.3 for the non-matroidal treatment of isolated vertices in ordinary graphs.

## 5. Hypercube interpretations for matroid automorphisms

It is possible to obtain geometric interpretations for each of our matroid automorphisms. What does it mean to 'interpret' a matroid operation geometrically? Formally, this involves group homomorphisms between $W$ and Aut($M_W$) for $W = O_n$ and $O_n^+$. We state the results more informally in this section to emphasize the nature of the correspondence. Since we have already realized the matroid structure via signed graphs, we now make the connection between the signed graph operations corresponding to matroid automorphisms and symmetries of $Q_n$.

We first recall a few standard facts about hypercubes. The hypercube $Q_n$ has $2^n$ vertices and $2n$ (maximal-dimensional) facets, where each vertex is incident with exactly $n$ facets. Fixing a vertex $v$, we arbitrarily label the $n$ incident facets $1, 2, \ldots, n$. Now each facet $i$ has an *opposite* or *antipodal* facet, which we label $i^*$. Then the $n$ facets incident to the antipodal vertex $v^*$ are labelled $1^*, 2^*, \ldots, n^*$, and all of the facets of $Q_n$ are labelled.

Alternatively, fixing $Q_n$ in $\mathfrak{R}^n$ as $[-1, 1] \times \cdots \times [-1, 1]$, the centroid of a facet has coordinates $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$. If the unique nonzero coordinate of the centroid occurs in position $i$, label this facet $i$ or $i^*$ according to whether the nonzero coordinate is 1 or $-1$. The convex hull of these centroids is the hypercube's dual, the *hyperoctahedron* or *cross polytope* [4]. Note that there is a one-to-one correspondence between the $2^n$ vertices of $Q_n$ and the $2^n$ possible labellings of the $n$ facets adjacent to a vertex (precisely one facet of each pair $i, i^*$ will be incident to the given vertex). Thus, the facet labelling around the vertex $w$ completely determines $w$.

We also review the structure of the symmetry groups $S_n, O_n^+$ and $O_n$. A brief account with many detailed references can be found in [5]. If $v$ is a vertex of $Q_n$, recall $\mathrm{stab}(v) = \{\varphi \in O_n | \varphi(v) = v\}$.

**Proposition 5.1.** $\mathrm{stab}(v) \cong S_n$.

**Proof.** Choose a vertex $v$ of $Q_n$ with incident facets $1, 2, \ldots, n$. A reflection through $v$ will fix $v$ and interchange precisely two of these facets, say facets $i$ and $j$. Representing this reflection by the permutation $(ij)$ immediately gives $S_n \leqslant \mathrm{stab}(v)$. But every symmetry $\varphi$ which fixes $v$ must permute the $n$ facets incident to $v$, and this permutation completely determines $\varphi$. Thus $\mathrm{stab}(v) \cong S_n$. □

When $\Gamma_W = K_n(A, B)$ for $n > 2$, we interpret the partition $(A, B)$ as follows. If $A = \{1, \ldots, k\}$ and $B = \{k+1, \ldots, n\}$, we find a vertex $v$ of $Q_n$ whose incident facets are labelled $1, \ldots, k, (k+1)^*, \ldots, n^*$. Then $\mathrm{stab}(v) \cong S_n \cong \mathrm{Aut}(M_W)$, and the matroid automorphisms of $M_W$ (which correspond to vertex permutations of $K_n(A, B)$ by Proposition 4.5(1)) correspond precisely to the $n!$ facet permutations of $\mathrm{stab}(v)$.

For $\Gamma_W = \pm K_n$ and $n > 2$, we again interpret matroid automorphisms via the hypercube $Q_n$. $W$ is isomorphic to the symmetry group $O_n^+$, the group of symmetries of the *half-measure polytope* whose vertices are alternate vertices of the hypercube $Q_n$. Coxeter and Moser point out (on page 123 of [5]) that $O_n^+$ is also the automorphism group of a 3-dimensional configuration Homersham Cox described in 1891.

Recall that a *half-turn* $\mu \in O_n^+$ is a symmetry of $Q_n$ that interchanges exactly two pairs of facets of $Q_n$ and leaves the remaining facets fixed, that is, $\mu$ is represented by the facet permutation $(ii^*)(jj^*)$. The 'axis' of rotation is a subspace of codimension 2.

**Proposition 5.2.** *Let $v$ be a vertex of $Q_n$ and let $\varphi \in O_n^+$. Then $\varphi = \mu\pi$, where $\mu$ is a composition of half-turns and $\pi \in \mathrm{stab}(v)$, and this decomposition is unique.*

**Proof.** The decomposition of the semi-direct product $O_n^+ \cong \mathbf{Z}_2^{n-1} \rtimes S_n$ gives $\varphi = \mu\pi$

uniquely, where $\mu \in \mathbf{Z}_2^{n-1}$ and $\pi \in S_n$. We must show that our interpretation of the factors is valid.

Since half-turns correspond to facet permutations of the form $(ii^*)(jj^*)$, the subgroup $H$ generated by the half-turns in $O_n^+$ is abelian, and every element of $H$ has order 2. Further, the order of $H$ is $2^{n-1}$ since $\mu \in H$ implies $\mu$ is a product of an even number of terms of the form $(ii^*)$. Thus, $H \cong \mathbf{Z}_2^{n-1}$ and $H \triangleleft O_n^+$.

For $\mathrm{stab}(v) \cong S_n$, note that $\mathrm{stab}(v) \cap H =< e >$, since any nontrivial composition of half-turns will fix no vertex of $Q_n$. This gives the desired decomposition. □

Proposition 5.2 gives a concrete way to realize any arbitrary symmetry $\varphi = \mu\pi \in O_n^+$. Note that $\mu$ (a composition of half-turns) sends the fixed vertex $v$ to some other vertex which is coloured the same as $v$ in a bipartite representation of $Q_n$. We visualize this process as follows: $\varphi$ first permutes the facets incident to $v$ (via $\pi \in \mathrm{stab}(v)$), and then maps $v$ to $w$, where $w$ has the same sense as $v$. Algebraically, this is accomplished by conjugating $\pi$ by $\mu_{vw} \in H$, where $\mu_{vw}$ is the unique composition of half-turns that maps $v$ to $w$: $\varphi = \mu_{vw}^{-1}\pi\mu_{vw}$. Thus, $\varphi = \mu\pi = \mu_{vw}\pi\mu_{vw}$, since $\mu_{vw}^{-1} = \mu_{vw}$. Thus $\mu$ is the commutator $\mu = \mu_{vw}\pi\mu_{vw}\pi^{-1}$.

We can now interpret the matroid automorphisms of $M_{O_n^+}$ via the hypercube. First note that half-turns in $O_n^+$ correspond to cutsets in $\pm K_n$: a half-turn $(ii^*)(jj^*)$ corresponds to the cutset that separates vertices $i$ and $j$ from the rest of the vertices of $\pm K_n$. It is possible to generate all the cutsets of $\pm K_n$ in this way if and only if $n$ is odd, that is, if and only if the centre $Z(O_n^+)$ is trivial (Table 10 of [5]). (The antipodal map $A$, which corresponds to the facet permutation $(11^*)(22^*)\cdots(nn^*)$, is the only nontrivial element of $Z(O_n)$. $A \in O_n^+$ if and only if $n$ is even. In fact, $A$ is the product of all the half-turns in $H$ if and only if $n$ is even. When $n$ is odd, the product of all the half-turns is the identity.) Thus, when $n$ is odd, an arbitrary matroid automorphism corresponds to the product of a vertex permutation of $\pm K_n$ (which corresponds to a facet permutation $\pi \in \mathrm{stab}(v)$ in the hypercube) and a cutset interchange of pairs of parallel edges (which corresponds to the composition of half-turns $\mu \in H$ in the hypercube) (by Proposition 4.5(2)).

When $n$ is even, the half-turn subgroup $H$ gives only half of the $2^{n-1}$ cutsets, and the geometric interpretation for the matroid automorphisms given above for odd $n$ does not work. In this case, we use the coordinate reflections (with reflecting hyperplanes $x_i = 0$) to generate all the cutsets (see below). Then an arbitrary automorphism of the matroid will still correspond to a product of a vertex permutation of $\pm K_n$ (which again corresponds to a facet permutation $\pi \in \mathrm{stab}(v)$) and a cutset interchange of pairs of parallel edges (which now corresponds to the composition of coordinate reflections).

The proof of the next proposition is similar to that of Proposition 5.2.

**Proposition 5.3.** *Let $v$ be a vertex of $Q_n$ and let $\varphi \in O_n$. Then $\varphi = \mu\pi$, where $\mu$ is a composition of coordinate reflections and $\pi \in \mathrm{stab}(v)$, and this decomposition is unique.* □

We now obtain the geometric interpretation for the matroid automorphisms of $M_{O_n}$ in much the same way we did for $M_{O_n^+}$, with the normal subgroup $\mathbf{Z}_2^n$ generated by the coordinate reflections ($ii^*$) (replacing the half-turn subgroup $H$) and no restriction on the image vertex $w$. Since the coordinate reflections that send $v$ to $w$ correspond to the same cutset as the reflections that send $v$ to the antipode of $w$, this procedure doubly counts the matroid automorphisms. The group-theoretic interpretation for this double-counting is $\mathrm{Aut}(M_W) \cong \mathrm{Inn}(W)$, the inner automorphisms of $W$, since the centre $Z(W) = \{e, A\}$, where $A$ is the antipodal map.

## 6. Zonotopes and symmetry groups

Zonotopes are projections of hypercubes [1, 18]; If $N$ is any $r \times k$ matrix, then the zonotope $Z(N)$ is a polytope in $\Re^r$ given by

$$Z(N) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{c}_i : -1 \leqslant \lambda_i \leqslant +1 \right\},$$

where $\mathbf{c}_i$ is the $i$th column of $N$.

Zonotopes are an important and well-studied class of polytopes with applications to oriented matroids, tiling problems and more. A striking example is the following result of Shephard and McMullen. A zonotope $Z$ tiles $\Re^n$ if and only if $M_Z$ is a binary matroid (where $M_Z$ is the linear dependence matroid of the columns of $N$ [12]). See Chapter 7 of [18] for more detail on these applications.

Our motivation is a bit more roundabout here. Beginning with a polyhedron, we form a related zonotope via the matrix of normal vectors. This allows us to relate the symmetry groups of two polyhedra, the original and the derived zonotope. In all cases, there are close connections between these two groups.

Thus we are interested in $Z(N_W)$ for the various matrices $N_W$ of normal vectors arising from the reflection subgroups $W$. When $W$ corresponds to $K_n$, the zonotope is the *permutahedron*; when $W$ corresponds to $\pm K_n^\circ$, the zonotope is a truncated version of the permutahedron. When $n = 3$, these zonotopes are Archimedean solids: the permutahedron is a truncated octahedron and the truncated permutahedron is a rhombitruncated cuboctahedron (see Figure 4). We also consider $Z(N_W)$ when $W$ is a reflection subgroup of $\mathrm{Sym}(P)$ for other polytopes $P$.

Our fundamental question again addresses the connection between $W$ and another symmetry group.

**Question 3.** What is the relationship among the reflection subgroup $W$, the matroid automorphism group $\mathrm{Aut}(M_W)$ and the symmetry group of the zonotope $\mathrm{Sym}Z(N_W)$?

We provide some partial answers to this question below. We begin with an example.

**Example 1 (Zonotopes for the Platonic solids).** We consider the symmetry groups of $Z(N_W)$, where $W$ ranges over all possible reflection subgroups of the symmetry groups of the Platonic solids. Since the symmetry group of a solid and its dual are the same, there are three symmetry groups to consider.

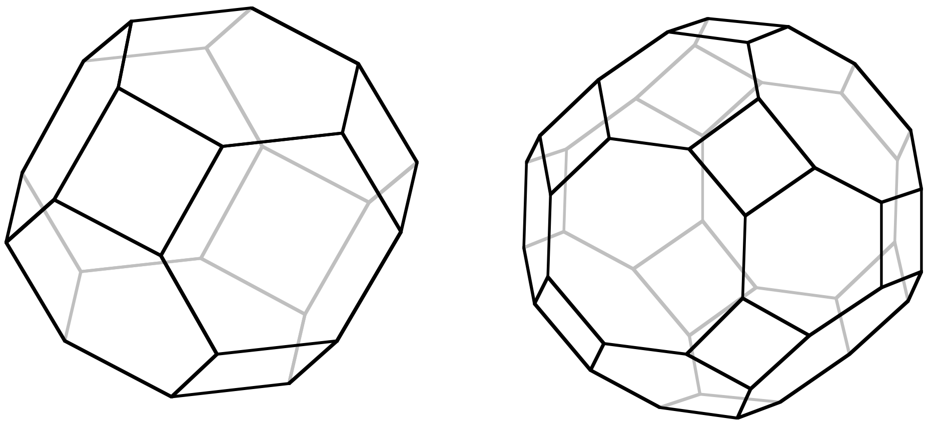

*Figure 4* Permutahedron (left) and truncated permutahedron (right)

*Table 1*

| | Reflections | Subgroup | Zonotope | Symmetry group |
|---|---|---|---|---|
| 1 | all 6 | $S_4$ | permutahedron | $S_4 \times \mathbf{Z}_2$ |
| 2 | $(ij), (jk), (ik)$ | $S_3$ | hexagon | $D_6$ |
| 3 | $(ij), (kl)$ | $\mathbf{Z}_2 \times \mathbf{Z}_2$ | square | $D_4$ |
| 4 | $(ij)$ | $\mathbf{Z}_2$ | segment | $\mathbf{Z}_2$ |

**Tetrahedron:** symmetry group $\cong S_4$;
**Cube or octahedron:** symmetry group $\cong \mathbf{Z}_2^3 \rtimes S_3 \cong S_4 \times \mathbf{Z}_2$;
**Dodecahedron or icosahedron:** symmetry group $\cong A_5 \times \mathbf{Z}_2$.

For each of these symmetry groups, we list the possible reflection subgroups whose normal vectors give the matrices that generate the associated zonotope. We also remark that the matroid automorphism groups $\mathrm{Aut}(M_W)$ for the subgroups of the tetrahedron and cube symmetry groups are determined by Theorem 4.6.

**Tetrahedron**

Our full generating matrix of normal vectors is

$$T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}.$$

There are four different nontrivial subgroups generated by reflections (from Section 2): $S_4, S_3, \mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2$. See Table 1 for the associated zonotopes and their symmetry groups.

*Table 2*

| | Reflections | Subgroup | Zonotope | Symmetry group |
|---|---|---|---|---|
| 1 | all 9 | $S_4 \times \mathbf{Z}_2$ | truncated permutahedron | $S_4 \times \mathbf{Z}_2$ |
| 2 | $(11^*), (22^*), (33^*)$ | $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ | cube | $S_4 \times \mathbf{Z}_2$ |
| 3 | 2 $\perp$ reflections | $\mathbf{Z}_2 \times \mathbf{Z}_2$ | square | $D_4$ |
| 4 | 1 reflection | $\mathbf{Z}_2$ | segment | $\mathbf{Z}_2$ |
| 5 | 3 reflections through a point | $S_3$ | hexagon | $D_6$ |
| 6 | $(ii^*), (jj^*), (ij)(i^*j^*), (i^*j)(ij^*)$ | $D_4$ | octagon (not regular) | $D_4$ |
| 7 | all except $(11^*), (22^*), (33^*)$ | $S_4$ | permutahedron | $S_4 \times \mathbf{Z}_2$ |

**Cube or octahedron**

The generating matrix of normal vectors is

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

This time, there are seven different subgroups generated by reflections (from Section 4). See Table 2 for the associated zonotopes and their symmetry groups.

We examine line 7 of Table 2 in more detail. The generating reflections correspond to the Puff signed graph $\pm K_3$, i.e., this subgroup is $O_3^+$. The matroid $M_{O_3^+} \cong M(K_4)$, the cycle matroid on $K_4$, giving $W \cong S_4$ by Theorem 4.6(2). This does not happen again; $W \ncong S_n$ for any $n \neq 4$ (for this class of subgroups $W$). (This is because the convex hull of the alternate vertices of a hypercube do not form a simplex in dimensions higher than 3.)

**Dodecahedron or icosahedron**

There are 15 reflections in this symmetry group. A generating matrix of normal vectors is

$$D = \begin{bmatrix} \tau & 0 & 0 & -\tau^2 & \tau^2 & \tau^2 & \tau^2 & \tau & \tau & -\tau & \tau & 1 & -1 & 1 & 1 \\ 0 & \tau & 0 & 1 & 1 & -1 & 1 & -\tau^2 & \tau^2 & \tau^2 & \tau^2 & \tau & \tau & -\tau & \tau \\ 0 & 0 & \tau & \tau & -\tau & \tau & \tau & 1 & -1 & 1 & 1 & \tau^2 & \tau^2 & \tau^2 & -\tau^2 \end{bmatrix},$$

where $\tau = (1+\sqrt{5})/2$ is the golden mean. Then each normal vector is parallel to some edge of the icosahedron. See [4] for more details on coordinates of Platonic (and other) solids.

Rather than list all possible subgroups of reflections, we concentrate on two examples. First consider edges $\{e_1, \ldots, e_5\}$ of the icosahedron which bound a regular pentagon, and let $H$ be the subgroup of $A_5 \times \mathbf{Z}_2$ generated by reflections in the five planes with normal vectors $\{e_1, \ldots, e_5\}$. Then, $H \cong D_5$, the symmetry group of a regular pentagon. Then, in analogy with the subgroup of line 2 of Table 1 (in which the three reflecting planes bound an equilateral triangle and the zonotope is a hexagon), we get that $Z(N_H)$ is a regular decagon, with symmetry group $D_{10}$.

As a second example for the icosahedron, let $H = A_5 \times \mathbf{Z}_2$, the full symmetry group. Then $Z(N_H)$ is the *rhombitruncated icosadodecahedron*, another Archimedean solid. See

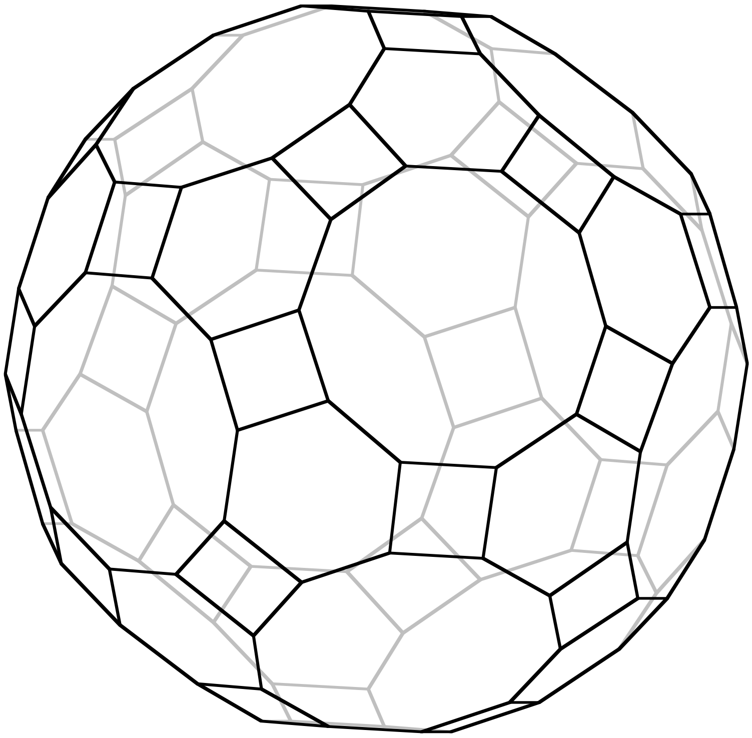

*Figure 5* Zonotope for icosahedron

Figure 5 for a drawing of this solid, which has 15 reflecting planes. Then $\mathrm{Sym}(Z(N_H)) \cong H \cong A_5 \times \mathbf{Z}_2$.

Note that in every case in Example 1, the number of vertices of the zonotope $Z(N_H)$ is the size of the subgroup $H$. We also note that $Z(N_H)$ always has at least as much symmetry as $H$. This motivates our concluding proposition.

**Proposition 6.1.** *Let $Z$ be the zonotope associated with the hypercube $Q_n$. Then* $\mathrm{Sym}(Z) \cong O_n$.

**Sketch of proof.** The truncated permutahedron has $2^n n!$ vertices, and each vertex has coordinates of the form $(\alpha_1 r_1, \ldots, \alpha_n r_n)$, where each $\alpha_i = \pm 1$ and $(r_1, \ldots, r_n)$ is some permutation of $(1, 3, 5, \ldots, 2n-1)$. We arrange these vertices into 'clumps:' Two vertices $(\alpha_1 r_1, \ldots, \alpha_n r_n)$ and $(\beta_1 s_1, \ldots, \beta_n s_n)$ of the truncated permutahedron are in the same clump if $\alpha_i = \beta_i$ for all $i$.

If vertices $v$ and $w$ are in the same clump, then there is a sequence of vertices $v = u_0, u_1, \ldots, u_{k-1}, u_k = w$ with distance $d(u_i, u_{i+1}) = 2\sqrt{2}$ for all $0 \leqslant i < k$. Further, if two vertices are in different clumps, then they cannot be $2\sqrt{2}$ units apart. Thus, the $2^n$ clumps correspond precisely to the $2^n$ vertices of the hypercubes, so any isometry of the hypercube is an isometry of the truncated permutahedron, and *vice versa.* □

In 3 dimensions, this proposition says that the cube and the rhombitruncated cuboctahedron have the same symmetry groups (see Figure 4).

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