# USING GEOMETRY IN TEACHING GROUP THEORY

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Abstract. Discrete groups of isometries provide a rich class of groups for detailed study in an undergraduate course in abstract algebra. After developing some geometric tools, we illustrate several important ideas from group theory via symmetry groups. In particular, we consider conjugation, homomorphisms, normal and nonnormal subgroups, generators and relations, quotient groups and direct and semidirect products. Understanding of these topics can be enhanced using symmetries of frieze and wallpaper patterns. We describe several examples in detail and indicate how other geometric investigations could be undertaken. We also indicate how four commercial software packages (*The Geometer's Sketchpad*, *KaleidoTile*, *TesselMania* and *Kali*) can be used to enhance students' understanding.

## 1. Introduction

The history of group theory is closely tied to measuring the symmetry of some object. Many popular algebra texts now include a section, chapter or collection of chapters that emphasize this approach. See [2], [8] and [16] for typical modern treatments of the subject. I believe the renewed interest in the origins of the subject stems (in part) from a shift in the way we teach our undergraduates—there are more "hands-on" activities and computer based projects than there were 5 years ago.

As a simple example of how symmetry operations can enhance understanding, consider the following standard homework exercise (4.6 in [2]):

*If $x, y$ and $xy$ all have order 2 in a group $G$, then show $xy = yx$.*

This is very easy to prove directly, but a visual approach may be illuminating for students. Consider the Euclidean plane and let $x$ correspond to the operation of a reflection in the $x$-axis and $y$ correspond to a reflection in the $y$-axis. Then it is obvious to students that the composition $xy$ corresponds to a $180°$ rotation about the origin. Clearly $x, y$ and $xy$ all have order 2; the conclusion that $xy = yx$ now follows because $180° = -180° \mod 360°$.

Although this approach does not lead to a formal proof of the exercise, it allows students to visualize the exercise concretely. This can mean more to them than the traditional symbol-pushing approach. It also may motivate them to discover other, related results on their own. (For example, if every nonidentity element of a group has order 2, then the group is abelian.)

## 2. Fundamental Properties of Isometries

We begin by reminding the reader of several basic results concerning isometries.

**Definition 1.** An *isometry* of $\mathbb{R}^n$ is a bijection that preserves distances. More precisely, $f$ is an isometry if, for any pair of points $P$ and $Q$ in $\mathbb{R}^n$, $d(P, Q) = d(f(P), f(Q))$.

It is worthwhile to explain to a class immediately why the set of all isometries forms a group. The group of isometries of the plane is usually called the *Euclidean* group. Also, one should have the students show that the four kinds of isometries of the plane satisfy this definition, i.e., show that translation, rotation, reflection and glide reflection are all isometries.

Working in 1 dimension (instead of 2) can be a beneficial starting point for students. It's easier to visualize the isometries and it allows students to build some confidence and intuition. In particular, it is a straightforward but useful exercise to prove that all isometries $f : \mathbb{R} \to \mathbb{R}$ have the form $f(x) = \pm x + b$, where $b \in \mathbb{R}$.

A more important classification for the remainder of this paper concerns the isometries of the plane. The next result is central.

**Proposition 1.**

*A. Every isometry of the plane is either a translation, rotation, reflection or glide reflection.*

*B. Every isometry of the plane is the composition of at most three reflections.*

Proofs of these two results can be found in [6]. Although it is probably not necessary to present or have students develop careful proofs of this proposition in an algebra course, I believe it is *very* important for students to develop geometric intuition by working with isometries. One way to lead students to a discovery of the proposition (with or without a careful proof as a goal) is to use the following exercise.

**Exercise 1** (Isometry basics)**.**

1. Given two congruent line segments $\overline{AB}$ and $\overline{A'B'}$, use *The Geometer's Sketchpad* to construct many different isometries, each of which maps $A \mapsto A'$ and $B \mapsto B'$.
2. Show that there are only two distinct isometries that map $A \mapsto A'$ and $B \mapsto B'$.
3. Show that each of these two distinct isometries can be written as a product of at most three reflections.
4. Now if $f$ is any isometry of the plane, show that $f$ is the product of at most three reflections.
5. By investigating the various possibilities (either in *The Geometer's Sketchpad* or by hand), show that the composition of 0, 1, 2 or 3 reflections is always a translation, rotation, reflection or glide reflection.

A few comments on this exercise are in order. Part 2 is believable for students using *The Geometer's Sketchpad*—different students may come up with isometries that appear distinct, but by following the image of a scalene triangle under apparently different maps, they are quickly convinced that there are only two possibilities. Further, part 4 follows from parts 2 and 3 since any isometry $f$ must map the line segment $\overline{AB}$ to the congruent segment $\overline{f(A)f(B)}$, where $A$ and $B$ are any two distinct points in the plane.

It is now possible to ask students if various subsets of isometries form a group. This involves checking the subsets for closure and inverses. As a sample, consider the following example.

**Exercise 2** (Group closure)**.** For each of the following sets $S$ of isometries of the plane, determine whether or not $S$ is a group.

1. $S = \{T \mid T \text{ is a translation}\}$.
2. $S = \{R \mid R \text{ is a reflection}\}$.
3. $S$ consists of all translations and all rational rotations, i.e., rotations through angles that are rational multiples of $\pi$.
4. $S$ consists of all isometries that fix the origin.
5. $S$ consists of all isometries that fix the $x$-axis.
6. $S$ consists of all isometries that fix the $x$-axis or the $y$-axis.

Problems 4 and 5 are important when we classify the discrete groups that fix a point (the cyclic and dihedral groups) and those that fix a line (the frieze groups). I like these exercises because they use geometry to illustrate group closure. They also show students that composing isometries is important, which is the point of the next section.

## 3. Three Important Shortcuts

In order to attack serious questions involving groups, it is necessary to understand two important group-theoretic operations as they relate to isometries: composition and conjugation. We begin by fixing notation for each of the four isometries of the plane:

1. Translation by a vector $\overline{v}$: $T_{\overline{v}}$
2. Rotation with center $P$ through an angle $\theta$: $r_{P,\theta}$
3. Reflection through a line $l$: $R_l$
4. Glide reflection along the *fixed* vector $\overline{v}$: $g_{\overline{v}}$

We also fix a symbol for each of these four isometries. See Figure 1.

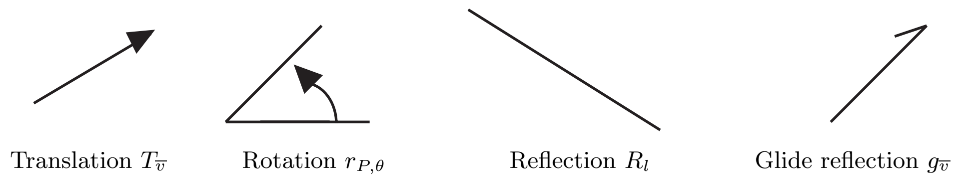



Figure 1.

Given two isometries, how can we easily determine what isometry corresponds to their composition? We now describe two shortcuts for determining the composition of two isometries, both of which involve the homomorphic images of the isometries. Our first shortcut is the observation that translations and rotations preserve sense, while reflections and glide reflections reverse sense. More precisely, we make the following definition.

**Definition 2.** An isometry $f$ is *direct* if it preserves right half planes, i.e., if $L$ is a directed line with right half plane $H$, then $f(H)$ is the right half plane for the line $f(L)$. Otherwise, $f$ is *indirect*.

Direct isometries are sometimes called *sense-preserving*, while indirect isometries are *sense-reversing* or *opposite*. In Figure 2, the left-most drawing represents a right-half plane, the center drawing is the image of this right-half plane under a direct isometry (in this case, a rotation), and the right-most drawing is the image of the same right-half plane under an indirect isometry (in this case, a reflection through a line perpendicular to the directed line).

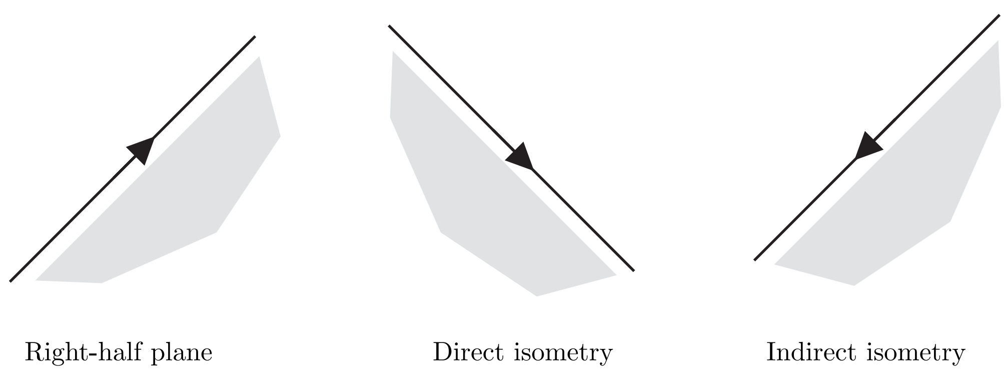



Figure 2.

We now state our first shortcut for composing isometries. The proof follows immediately from the definition.

**Proposition 2.**

1. *The composition of two direct isometries is direct.*
2. *The composition of a direct and an indirect isometry is indirect.*

| $*$ | $D$ | $I$ |
|---|---|---|
| $D$ | $D$ | $I$ |
| $I$ | $I$ | $D$ |

Table 1. Composing direct and indirect symmetries

3. *The composition of two indirect isometries is direct.*

This cuts down on the possibilities for the students to check when composing isometries, especially when used in conjunction with our next shortcut. The above proposition is summarized in Table 1, where $D$ stands for a direct isometry and $I$ stands for an indirect one.

Students recognize the similarity between this table and the multiplication table for $\mathbb{Z}_2$. (See Exercise 9 in Chapter 1 of [8].) This suggests that a homomorphism is lurking in the background (which, of course, it is). This can be a starting point for the topic of homomorphisms, or simply an easy example.

Our second shortcut involves recording the effect a given isometry $f$ has on a family of parallel, directed lines. Let $\mathcal{L}_{\overline{u}}$ denote the set of all lines parallel to a given unit vector $\overline{u}$, directed as $\overline{u}$ is. Clearly, this family is carried to another family of parallel, directed lines, say $\mathcal{L}_{\overline{w}}$. Thus, while $f$ itself may not map $\overline{u}$ to $\overline{w}$, this isometry *induces* a map on unit vectors, i.e., $f$ induces a mapping of the unit circle to itself. (This time, the induced map is the image of a homomorphism that sends the full Euclidean group to the isometries that fix the unit circle. The kernel of this map is the normal subgroup of all translations.)

The two most important properties of this shortcut are summarized in the next proposition.

**Proposition 3.** *Let $f$ be a direct isometry, $\mathcal{L}_{\overline{u}}$ denote a family of parallel, directed lines and $\mathcal{L}_{\overline{w}}$ be the image of $\mathcal{L}_{\overline{u}}$ under $f$, where $\overline{u}$ and $\overline{w}$ are unit vectors.*

1. *$f$ is a translation iff $\overline{u} = \overline{w}$.*
2. *If $f$ is a rotation, then the angle between $\overline{u}$ and $\overline{w}$ is the angle of the rotation.*

We now give an example of how these two shortcuts can be used to determine the composition of two isometries.

**Proposition 4.**

1. *$T_{\overline{v}}\ r_{P,\theta} = r_{Q,\theta}$ for some point $Q$.*
2. *$r_{P,\theta_1}\ r_{Q,\theta_2} = r_{R,\theta_1+\theta_2}$ for some point $R$, unless $\theta_1 + \theta_2$ is a multiple of $2\pi$, in which case $r_{P,\theta_1}\ r_{Q,\theta_2} = T_{\overline{v}}$ for some vector $\overline{v}$.*
3. *$R_{l_1} R_{l_2} = r_{P,\theta}$, unless $l_1$ and $l_2$ are parallel, in which case $R_{l_1} R_{l_2} = T_{\overline{v}}$.*

*Proof.*

1. By shortcut 1, the composition of a translation and a rotation is a direct isometry. But this composition does not fix a family of parallel, directed lines, so by shortcut 2, $T_{\overline{v}} r_{P,\theta} = r_{Q,\theta}$ for some point $Q$.
2. By shortcut 1, $r_{P,\theta_1} r_{Q,\theta_2}$ is direct. This composition will leave a family of parallel, directed lines invariant iff $\theta_1 + \theta_2$ is a multiple of $2\pi$. Proposition 3 now gives the result.
3. Again, shortcut 1 tells us this composition is direct. The result now follows as in part 2.

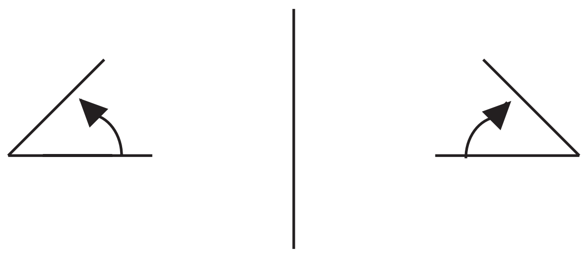
□

We point out that the usual proofs of parts 1 and 2 of this proposition are much more involved geometrically (although a geometric proof will determine the point $Q$ in part 1 and the point $R$ in part 2). Part 3 is easy to prove directly. It is fun to demonstrate part 3 with *The Geometer's Sketchpad*. To do this, draw a scalene triangle and two lines. By making the two lines mirrors, one can reflect the triangle in each line (and hide the intermediate reflection if desired). Then the dynamic nature of the program allows one to vary the angle between the two lines and watch the doubly reflected image of the triangle move. Furthermore, the angular speed with which the triangle moves is twice the angular speed of the mirror line. This leads to the conclusion that the angle of rotation is twice the angle between the mirror lines. One can also repeat this experiment when the two lines are parallel.

Our final shortcut involves conjugation of isometries and our symbols from Figure 1. We omit the straightforward but long proof.

**Proposition 5.** *Let $f$ and $g$ be isometries. Then the symbol for the isometry $fgf^{-1}$ is obtained by applying the isometry $f$ to the symbol corresponding to $g$.*

As an example of this result, let $f = R_l$ and $g = r_{P,\theta}$, as in Figure 3. Then the conjugate isometry $R_l\, r_{P,\theta}\, R_l^{-1}$ is the rotation $r_{P',-\theta}$, where $P'$ is the image of the point $P$ when reflected through the line $l$.



$r_{P,\theta}$ $R_l$ $R_l r_{P,\theta} R_l^{-1}$

Conjugation of a rotation by a reflection.

FIGURE 3.

Proposition 5 is very useful for determining if a given subgroup is normal. We conclude this section with an entertaining application, due to Tom Brylawski (private communication).

**Exercise 3.** Let $A, B$ and $C$ be the vertices of a triangle $T$ in the plane. Now perform the following operations on $T$:

1. Rotate $T$ 180° about $A$; call the resulting triangle $T_1$, with vertices $A_1(= A), B_1$ and $C_1$.
2. Rotate $T_1$ 180° about $B_1$; call the resulting triangle $T_2$, with vertices $A_2, B_2(= B_1)$ and $C_2$.
3. Rotate $T_2$ 180° about $C_2$; continue the labeling process as above.
4. Rotate $T_3$ 180° about $A_3$.
5. Rotate $T_4$ 180° about $B_4$.
6. Rotate $T_5$ 180° about $C_5$.

Show that $T_6 = T$, i.e., the triangle returns to its original position.

To do this exercise, we first let $H_P$ denote the 180° rotation (or *half turn*) about the point $P$. Then the isometry represented by the first step above is simply $H_A$. After 2 steps, the isometry is the composition $H_{B_1}H_A$, using the usual right-to-left ordering on composition. But $H_{B_1} = H_A H_B H_A^{-1}$ by our conjugation shortcut of Proposition 5. Thus

$H_{B_1}H_A = (H_A H_B H_A^{-1})H_A = H_A H_B$. The net effect of conjugation is that *multiplication has been reversed.*

A similar argument shows that, after three steps, the isometry is given by the composition $H_A H_B H_C$. Now note that $H_A H_B H_C = H_D$ for some point $D$, since this composition is direct (by shortcut 1) and it reverses any family of parallel, directed lines (shortcut 2). Thus, after 6 steps, the isometry is represented by $(H_D)^2$, which is the identity. This establishes the result.

We remark that other proofs are possible here, but this is probably the quickest. Our proof also makes use of all three of the shortcuts introduced in this section. It is a good exercise to carry out the 6 steps of the exercise either by hand (with a triangle cut out of paper) or via *The Geometer's Sketchpad.* Students can then guess the answer before seeing a proof.

## 4. Subgroups of Crystallographic Groups

We now illustrate several topics typically presented in a course that introduces groups.

**Example 1** (Generators and relations)**.** Let $G$ be the symmetry group of an infinite strip of V's, i.e., the symmetry group of the pattern

$$\cdots \quad \text{V} \quad \text{V} \quad \text{V} \quad \text{V} \quad \text{V} \quad \text{V} \cdots$$

This symmetry group, the infinite dihedral group $D_\infty$, is one of the seven *frieze* groups. See [15] for a detailed presentation of these seven groups. This group is generated by two adjacent vertical reflections, one that is halfway between two adjacent V's and the other that bisects a V. Call these reflections $R_1$ and $R_2$, respectively, as in Figure 4. The collection of all lines of reflection divides the plane into a family of vertical, parallel strips. Note also that the collection of all reflections in $G$ fall into two conjugacy classes (by shortcut 3, the conjugation rules). The reflections conjugate to $R_1$ are precisely the reflections that are halfway between adjacent V's, while the reflections conjugate to $R_2$ are the reflections through the V's.

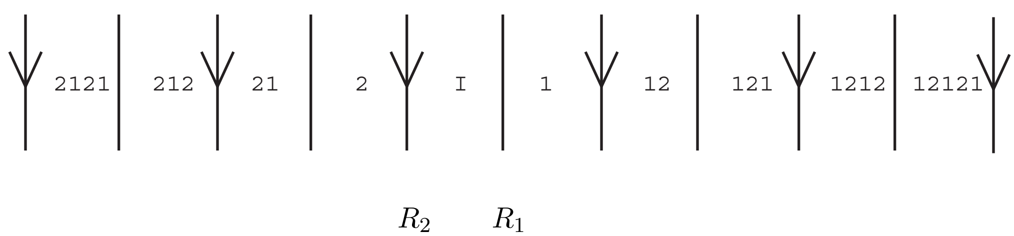



Figure 4.

Now label the strip between the two parallel lines corresponding to $R_1$ and $R_2$ with the identity $I$, and we call this the *fundamental region.* Then we can label each strip uniquely as follows: Let $S$ be a vertical strip between adjacent reflections and assume inductively that the strips between $I$ and $S$ have already been labeled. Assume $S$ is to the right of $I$ and let $S'$ denote the strip that borders $S$ on the left. (If $S$ is to the left of $I$, let $S'$ denote the strip bordering $S$ on the right.) Let $w$ be the label for $S'$ and let $R'$ be the reflection corresponding to the border between $S'$ and $S$. Then label $S$ by $R_i w$, where $i = 1$ or 2 depending on whether $R'$ is conjugate to $R_1$ or $R_2$. (This *reversed* multiplication stems from our conjugation rules—see Example 3.) In Figure 4, we write 1 for $R_1$, 121 for $R_1R_2R_1$, and so on.

This gives a labeling of each strip with some word of alternating $R_1$'s and $R_2$'s. This approach, which is dual (in the graph theoretic sense) to the *Cayley graph* of the group, is called the *dihedral kaleidoscope* and is presented for many reflection groups in Coxeter's classic [7].

We now use this example to illustrate several topics. The same analysis can be used on more complicated symmetry groups (e.g., the wallpaper groups). The payoff for working through harder examples is a deeper understanding of the topics involved. We concentrate on our simple example here to illustrate the main ideas; if some students or a class becomes especially interested in this approach, then it might be worthwhile to explore deeper examples.

**Topic 1** (Generators and relations)**.** The correspondence between the regions of Figure 4 and the words of $D_\infty$ shows that the reflections $R_1$ and $R_2$ generate $D_\infty$. The relations $R_1^2 = R_2^2 = I$ follow from the simple fact that when one crosses the line from region $I$ to region $R_1$, say, and then crosses the line back again to $I$, one's path corresponds to the word $R_1^2$, but the region occupied is $I$. This makes use of the fact that our region labeling scheme could easily be extended to *paths* from region $I$ to any other region. Thus two paths that begin in the same region and end in the same region must correspond to the same word in $G$. The "straight-line" or "light-ray" path will give a shortest or reduced word for the region. Longer paths that wind back and forth will give words that can be reduced via the relations to the shortest word.

**Topic 2** (Normal subgroups, cosets, quotient groups)**.** There are several ways to find normal subgroups of $D_\infty$. By our conjugation rules, the set of all translations in $D_\infty$ is a normal subgroup. (Note that conjugating a translation $T$ by either of the generators $R_1$ or $R_2$ yields $T^{-1}$. Thus the set of translations is invariant under conjugation.)

This gives two cosets in $D_\infty$, $\mathcal{T}$ and $R_1\mathcal{T}$, where $\mathcal{T}$ is the normal subgroup of translations. The coset $\mathcal{T}$ corresponds to all regions with labels using an even number of reflections (as these are precisely the translations), while the coset $R_1\mathcal{T}$ corresponds to all regions whose labels use an odd number of reflections.

Since there are only 2 cosets, the quotient group is $\mathbb{Z}_2$. To see this via the generators and relations, note that modding out by all translations forces $R_1R_2 = I$ in the quotient group. No additional new relations are forced, since $R_1R_2$ generates $\mathcal{T}$. Thus,

$$D_\infty/\mathcal{T} = \langle R_1, R_2 \mid R_1^2 = R_2^2 = R_1R_2 = I\rangle.$$

This easily reduces to $\langle R \mid R^2 = I\rangle$.

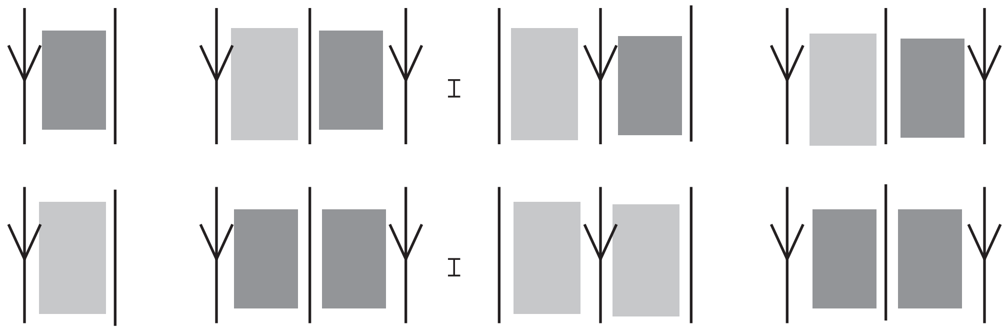
Other subgroups of translations are also normal subgroups of $D_\infty$. For example, if $H$ is the subgroup generated by $T^5$, where $T = R_1R_2$ is the shortest translation, then $H$ is normal (by our conjugation rules) and $D_\infty/H$ has presentation

$$D_\infty/H = \langle R_1, R_2 \mid R_1^2 = R_2^2 = (R_1R_2)^5 = I\rangle.$$

This is a standard presentation for the dihedral group $D_5$. This procedure is entirely analogous to the usual way quotient groups are used to obtain $\mathbb{Z}_5$ from $\mathbb{Z}$. It is an entertaining exercise to color the regions of Figure 4 with 10 colors corresponding to the 10 cosets generated.

To see how a nonnormal subgroup can be generated, let $H$ be the subgroup generated by $R_1R_2R_1$ and $R_2R_1R_2$. This subgroup will include every *third* reflection (as well as translations of the form $T^{3k}$, where $T = R_1R_2$ is the shortest translation as before). Note that $H \cong D_\infty$ since $H$ is generated by two parallel reflections. Then $H$ is not normal—$R_1$ is clearly conjugate to $R_2R_1R_2$, but $R_1 \notin H$.

In spite of this nonnormality, we can still attempt the *coset coloring* mentioned above. Then we get the left cosets $H, R_1H$ and $R_2H$. This induces the left coset coloring of Figure 5, where each region in a given coset is shaded the same.




Row one—Left coset coloring
Row two—Right coset coloring
FIGURE 5.

In the same way, we can shade the regions using the right cosets $H, HR_1$ and $HR_2$. These cosets do not coincide with the left cosets because $H$ is not normal. This nonnormality is reflected in the coloring—the right coset coloring does not match the left coset coloring.

We conclude by considering direct and semidirect products of groups. Our geometric examples for these topics again involve the frieze groups.

**Example 2.** Let $G$ be the symmetry group for the pattern

$$\cdots \quad \text{H} \quad \text{H} \quad \text{H} \quad \text{H} \quad \text{H} \quad \text{H} \cdots$$

Then $G$ is generated by two vertical reflections $R_1$ and $R_2$, as before, together with one horizontal reflection $R_h$. The collection of all vertical lines of reflection with the horizontal line partitions the plane into regions, as in Figure 4. Our labeling scheme again gives shortest words for each region and the fact that the line corresponding to $R_h$ is orthogonal to both of the lines for $R_1$ and $R_2$ means that $R_h$ commutes with both $R_1$ and $R_2$. Thus, we get the presentation

$$G = \langle R_1, R_2, R_h \mid R_1^2 = R_2^2 = R_h^2 = R_h R_1 R_h R_1 = R_h R_2 R_h R_2 = I \rangle$$

**Topic 3** (Direct product)**.** The presentation for the group $G$ from Example 2 shows $R_h$ commutes with the other generators. Thus $G$ is a direct product of the normal subgroup generated by $R_1$ and $R_2$ (which is isomorphic to $D_\infty$) and the normal subgroup generated by $R_h$ (which is isomorphic to the dihedral group generated by a single reflection, $D_1$). Thus

$$G \cong D_\infty \times D_1.$$

We conclude this section with another frieze group that has a semidirect product decomposition.

**Example 3.** This time, let $G$ be the symmetry group for the following pattern:

$$\cdots \cup \quad \cap \quad \cup \quad \cap \quad \cup \quad \cap \quad \cdots$$

Then $G$ contains vertical reflections (through the union and intersection symbols), half turns (half way in between the symbols) and horizontal glide reflections (and, of course, translations). If $R$ is a reflection and $H$ is a half-turn, then $RH$ is a glide reflection. (To see this, note $RH$ is indirect by shortcut 1, so $RH$ is either a reflection or a glide reflection. If $RH = R'$ for some reflection $R'$, then $RR' = H$. But $RR'$ is a translation since $R$ and $R'$ are parallel reflections. Thus $RH = g$ for some glide reflection $g$.) If $R$ is a vertical reflection and $H$ a half-turn whose center is as close as possible to the line corresponding to $R$, we get the following presentation for $G$:

$$G = \langle R, H \mid R^2 = H^2 = I \rangle.$$

Thus $G \cong D_\infty$. (In fact the seven frieze groups fall into only four isomorphism classes: $C_\infty, C_\infty \times D_1, D_\infty, D_\infty \times D_1$.)

**Topic 4** (Semidirect products)**.** Let $G$ be the symmetry group in Example 3 and let $N$ be the normal subgroup of $G$ generated by all vertical reflections. Then $N \cong D_\infty$. Let $R_1$ and $R_2$ be adjacent vertical reflections and $H$ the half-turn between these reflections. Then $HR_1H = R_2$, i.e., conjugation by $H$ interchanges the generators of $N$. Then $G/N = \langle H \mid H^2 = I\rangle \cong D_1$. Let $M = \{I, H\}$ be the (nonnormal) subgroup generated by $H$. This gives the semidirect product decomposition

$$G \cong N \rtimes M.$$

## 5. Conclusion

Almost all of the topics typically encountered in an introductory group theory course can be illustrated with symmetry groups. This is an engaging approach for many students because they see a connection between two areas (group theory and geometry) and get to try out their group theory ideas in a different setting. In addition, the patterns we study are taken from different cultures and times. For example, some of the most beautiful patterns are found in the Alhambra, the fourteenth century Moorish palace in Granada. It was after his trips to this palace in the 1920's and 1930's that Dutch artist M. C. Escher made a detailed study of the possible repeating patterns and developed his own classification scheme for them. See [18] for more information on Escher.

Many computer programs are available to illustrate these ideas. *The Geometer's Sketchpad* [13] is very easy to use and allows the user to define any of the standard isometries (translation, reflection and rotation). *TesselMania* [14] allows one to create one's own repeating patterns, and *Kali* [9] allows one to select an isometry group and then draw a picture that has the selected group as its symmetry group. *KaleidoTile* [10] allows visualization of a symmetry group generated by reflections, including hyperbolic groups and the finite symmetry groups in three dimensions. Links to each of these pieces of software are available from my page at this volume's web site. (See the appendix for details.)

Some suggestions for specific projects appear in [11], while [19] includes several computer-based projects for an algebra class. The bibliography also includes many interesting resources not explicitly referred to in the body of this paper.

## References

[1] C. Alexander, I. Giblin and D. Newton, *Symmetry groups of fractals*, The Math. Intelligencer, **14** (Spring 1992) 32-38.
[2] M.A. Armstrong, *Groups and Symmetry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1988.
[3] T. Brylawski, *Notes on Mathematics and Art*, unpublished.
[4] J.H. Conway, *The orbifold notation for surface groups*, in Groups, Combinatorics and Geometry, (Liebeck and Saxl, ed.) London Mathematical Society Lecture Note Series 165, Cambridge, 1992.
[5] H.S.M. Coxeter, *A simpler introduction to colored symmetry*, International Journal of Quantum Chemistry, **31** (1987) 455-461.
[6] H.S.M. Coxeter, *Introduction to Geometry*, second edition, John Wiley & Sons, Inc., 1961, 1969.
[7] H.S.M. Coxeter, *Regular Polytopes*, third edition, Dover, 1973.
[8] J. Gallian, *Contemporary Abstract Algebra*, fourth edition, Houghton Mifflin, 1998.
[9] The Geometry Center, *Kali*, `http://www.geom.umn.edu/software/download/kali.html`.
[10] The Geometry Center, *KaleidoTile*, `http://www.geom.umn.edu/software/download/kali.html`.
[11] G. Gordon, *Using wallpaper groups to motivate group theory*, PRIMUS **VI** (1996) 355-365.
[12] B. Grnbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman, 1989.
[13] Key Curriculum Press, *The Geometer's Sketchpad*, Emeryville, CA, `http://www.keypress.com/`.
[14] The Learning Company (formerly from MECC), *TesselMania*, Novato, CA, `http://www.mecc.com`.

[15] G.E. Martin, *Transformation Geometry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1982.
[16] V.V. Nikulin and I.R. Shafarevich, *Geometries and Groups*, Springer-Verlag, Berlin, 1987.
[17] D. Schattschneider, *The plane symmetry groups; their recognition and notation*, Amer. Math Monthly, **85** (1978) 439-450.
[18] D. Schattschneider, *Visions of Symmetry*, W. H. Freeman & Co., 1990.
[19] D. Schattschneider, *Visualization of group theory concepts with dynamic geometry software*, in Geometry Turned On, Math Notes **41**, MAA, 1997.
[20] D. Seymour and J. Britton, *Introduction to Tessellations*, Dale Seymour Pub., 1989.
[21] D.K. Washburn and D.W. Crowe, *Symmetries of Culture*, Univ. of Washington Press, Seattle, 1988.

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