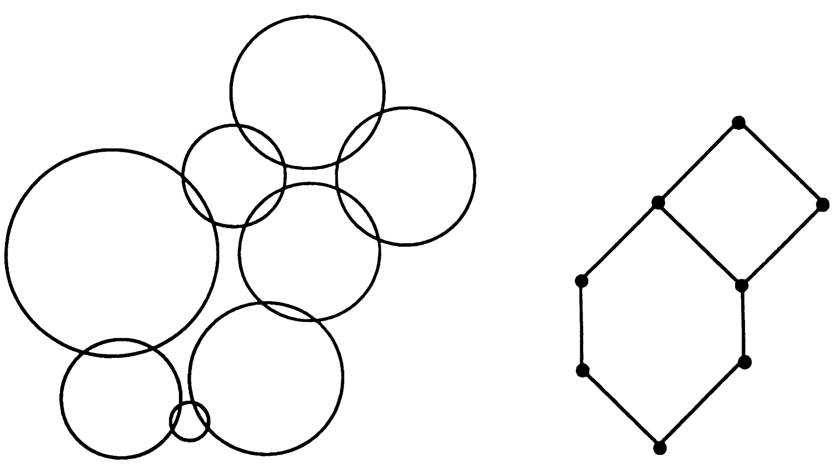
Workable Gears, Archimedian Solids and Planar Bipartite Graphs

Gary Gordon

1. INTRODUCTION. My young daughters have a popular children's game which involves setting up a system of interlocking gears. Each of the plastic gears can be fastened to a plastic board in a variety of places. The gears come in three sizes and can be adorned with colorful plastic animals and other decorations. This toy can hold their attention for a long time (where 'long time' is defined as longer than five minutes). We will call an arrangement of the gears (which consists of a finite number of gears, some pairs of which interlock and some which don't) *workable* if whenever any one gear is turned, all the gears turn freely. Children usually discover that it is quite easy to construct an unworkable arrangement of the gears.



Gear arrangement Corresponding graph

Figure 1

These gears also have an easy and direct connection to graph theory. (We will assume familiarity with most of the basics of graph theory. See [1], [3], [5] or [7] for very readable introductions which emphasize different aspects of the subject.) Given an arrangement S of the gears, define a graph $G(S)$ whose vertices correspond to the gears themselves. Define the edges of $G(S)$ so that uv is an edge joining vertices u and v if and only if the gears corresponding to u and v actually interlock. (See FIGURE 1, where the gears are drawn as circles with interlocked gears overlapping.) Recall that the chromatic number of a graph is the smallest number of colors needed to color the vertices of G so that adjacent vertices receive different colors. I occasionally give the following question as a homework problem in graph theory courses.

Problem A. *Let G $(= G(S))$ be a graph obtained from a workable arrangement of gears. What can you say about G? In particular, can you determine the chromatic number of G?*

Students realize quickly that one thing they can say is that G must be planar, that is, G can be drawn in the plane so that edges don't cross. It's not much harder to see that G will correspond to a workable arrangement of gears if and only if G has no odd cycles. (If the arrangement is workable, then it is clear that G has no odd cycles; the other direction can be established by induction on the number of gears, for example.) Since the students have already learned (or are about to learn) that a graph G has no odd cycles if and only if G is bipartite (i.e., G has chromatic number 2), they can prove the following proposition (and solve Problem A).

Proposition 1. *If G corresponds to a workable arrangement of gears, then G is a planar bipartite graph.*

The rest of this paper is devoted to constructing various planar bipartite graphs which are 'nice' in some way. All of the graphs we will consider are assumed to be connected. In Section 2, we discuss *regular* planar bipartite graphs, i.e., planar bipartite graphs in which every vertex has the same degree. Section 3 is concerned with *semi-regular* planar bipartite graphs, i.e., graphs in which all the red vertices have degree a and all the blue vertices have degree b for given positive integers a and b. These graphs will be closely related to the Archimedian solids. (In a bipartite graph, we will refer to the vertex partition induced by the 2-coloring by simply saying 'the red vertices' or 'the blue vertices.')

2. REGULAR PLANAR BIPARTITE GRAPHS. If G is a (connected) planar bipartite graph which is regular, then it is easy to show that the number of red vertices and the number of blue vertices must be equal.

Problem B. *Determine all possible positive integers r and n such that there is a planar bipartite graph G which is regular of degree r and which has n red vertices (and n blue vertices).*

The main tool used in solving all of the problems considered here is Euler's famous polyhedral formula, which he discovered around 1750. (See Theorem 8.1.1 of [**3**], for example.)

Theorem 2 (Euler's Polyhedral Formula). *If a plane drawing of a connected graph with v vertices and e edges has r regions, then $v - e + r = 2$. (This formula includes the unbounded region in the count for r.)*

The next corollary (which appears as Theorem 8.1.5 in [**3**]) is also a standard result. It follows from Euler's formula and the fact that each cycle in a bipartite graph contains at least four edges.

Corollary 3. *If G is a planar bipartite graph with $v \geq 3$ vertices and e edges, then $e \leq 2v - 4$.*

To solve Problem B, we begin by applying Corollary 3 to G. Let G be a planar bipartite graph which is regular of degree r on $2n$ vertices (n of which are in each of the two color classes). Then the number of edges is given by $e = rn$, so we immediately get $rn \leq 2(2n) - 4$, so $r < 4$.

It remains to investigate the cases $r = 1$, $r = 2$ and $r = 3$ separately. The determination of all possible values for n for the cases $r = 1$ and $r = 2$ is left to the reader (see Proposition 4). We will now determine all possible values of n for the case $r = 3$.

A graph which is regular of degree 3 is called *cubic*. The smallest cubic bipartite graph is $K_{3,3}$, the 'three houses and three utilities' graph. (The *complete bipartite graph*, denoted $K_{m,n}$, is the bipartite graph having m red vertices and n blue vertices with every red vertex adjacent to every blue vertex.) $K_{3,3}$ is not planar, so the smallest cubic planar bipartite graph must have $n \geq 4$.

Finding examples of cubic planar bipartite graphs is not hard. We now construct two classes of cubic planar bipartite graphs, one for even n and one for odd n. If $n \geq 4$ is even, define a graph B_n with vertices $\{1, 2, \ldots, 2n\}$ as follows: Form two n-cycles, one with vertices $\{1, \ldots, n\}$ and the other with vertices $\{n + 1, \ldots, 2n\}$, then add n edges by joining vertices k and $n + k$ for each k, $1 \leq k \leq n$. (B_4 is isomorphic to the graph associated with a three-dimensional cube.) For odd $n > 5$, define B_n by modifying B_{n-1} as follows: Delete edges $(1, n + 1)$, $(3, n + 3)$ and $(5, n + 5)$ from B_{n-1}, then add two new vertices x and y, joining vertex x to vertices 1, 3 and 5 and vertex y to vertices $n + 1$, $n + 3$ and $n + 5$. See FIGURE 2 for planar drawings of B_6 and B_7. The red vertices are labeled 1, the blue ones are labeled 2.

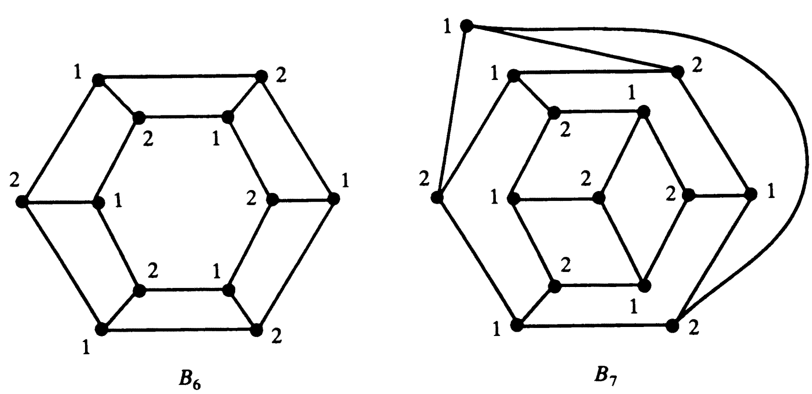


Figure 2

The above procedure fails when $n = 5$. In fact, there is no cubic planar bipartite graph with $n = 5$. If such a graph G existed, then it would have 10 vertices, 15 edges and 7 regions. For a region R, let $e(R)$ denote the number of edges which bound R. Then $e(R)$ is even (since G is bipartite) and $\Sigma e(R) = 30$ (since each edge is counted twice), where this sum extends over all 7 regions of G. Thus, exactly one of the regions is bounded by a 6-cycle and the six remaining regions are all bounded by 4-cycles. The reader can now show that G cannot be planar; we omit the remaining details.

The next proposition summarizes the results of this section and completely solves Problem B.

Proposition 4. *If G is a (connected) planar bipartite graph which is regular of degree r and has n vertices in each color class, then the only possibilities for n and r are the following*:

(a) $r = 1$ *and* $n = 1$,

(b) $r = 2$ *and* $n \geq 2$,

(c) $r = 3$ *and* $n \geq 4$, *with* $n \neq 5$.

Furthermore, each of these possibilities can be realized.

3. SEMI-REGULAR PLANAR BIPARTITE GRAPHS AND ARCHIMEDIAN SOLIDS. When we allow a little more flexibility in our planar bipartite graphs, we can get some more interesting examples. In this section we consider semi-regular planar bipartite graphs.

Problem C. *Determine all possible positive integers a and b (with $a < b$) such that there is a semi-regular planar bipartite graph G in which every red vertex has degree a and every blue vertex has degree b.*

Suppose G is a semi-regular planar bipartite graph with m red vertices, each of degree a and n blue vertices, each of degree b, where $a < b$. Then G has $v = m + n$ vertices and $e = am = bn$ edges. By Corollary 3, $am \leq 2m + 2n - 4$. Substituting $m = bn/a$ and simplifying gives the inequality

$$\frac{1}{a} + \frac{1}{b} \geq \frac{1}{2} + \frac{2}{e}. \qquad (\ddagger)$$

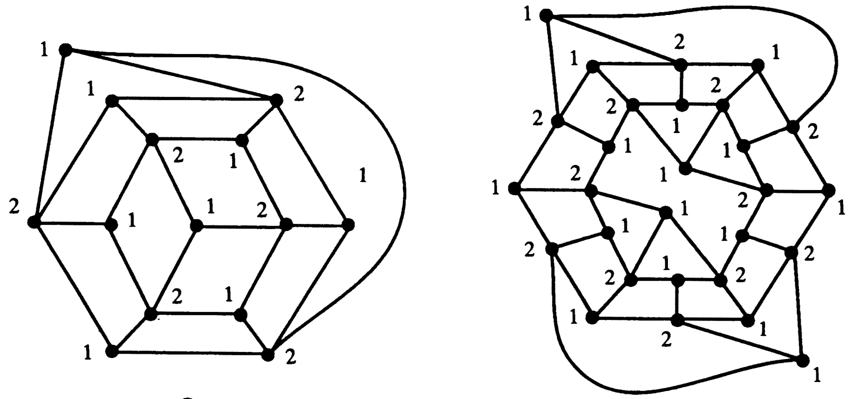
Thus, the only possible values for a and b are the following:

1. $a = 1$ and $b > 1$,
2. $a = 2$ and $b > 2$,
3. $a = 3$ and $b = 4$ or 5.

We again leave to the reader the construction of the appropriate graphs for the first two cases above and turn our attention to the case where $a = 3$ and $b = 4$. In the smallest possible example, the inequality in ($\ddagger$) will be replaced by equality. Solving for the number of edges then gives $e = 24$, so $m = 8$ and $n = 6$ (since $3m = 4n = e$). From Euler's formula, G must have 12 regions. Furthermore, each region of (any planar drawing of) G is bounded by exactly 4 edges. We can now construct G by modifying the graph B_6 constructed above (see Figure 2). As in the modification which produced B_7, we again add two new vertices and six new edges to B_6, but this time we don't delete any old edges (see FIGURE 3). We denote this graph by $C_{6,8}$.

We can create an infinite family of semi-regular planar bipartite graphs with $a = 3$ and $b = 4$ by modifying the graphs B_{6k} (for any $k \geq 1$) in an analogous way. Add $2k$ vertices to B_{6k} and join each of these new vertices to three 'blue' vertices of B_{6k}. We denote this family by $C_{6k,8k}$ for $k \geq 1$. See FIGURE 3 for a drawing of $C_{12,16}$, which is a modification of B_{12}.

Since we are considering semi-regular graphs, it is not surprising to discover a connection between the graphs we have constructed and another family of semi-regular graphs, the Archimedian solids. An *Archimedian*, or *semi-regular* solid has the property that its faces are all regular polygons, but of two or more kinds, all its vertices are identical and it can be circumscribed by a regular tetrahedron so that four of its faces lie on the four faces of the tetrahedron. (Dropping the last requirement concerning the circumscribed tetrahedron allows infinite families of prisms and antiprisms and the *pseudo rhombicuboctahedron*. See Figures 2.10 and

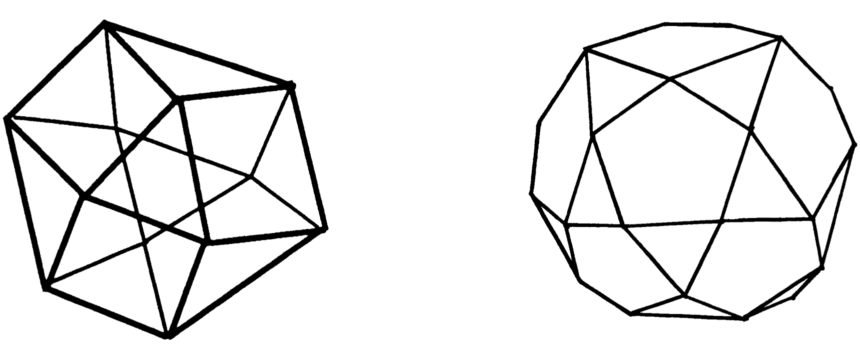


$C_{6,8}$ $C_{12,16}$

Figure 3

2.13 of [4].) There are 13 Archimedian solids, 11 of which can be obtained from the five Platonic solids by truncation. Archimedes' account of the 13 solids which bear his name is lost, presumably in the great fire of Alexandria. Heron writes that Archimedes ascribed the cuboctahedron to Plato. (See the historical notes in Chapter 2 of [2] and the Foreword of [6].) Kepler's book *The six-cornered snowflake*, published in 1609, includes what is believed to be the first complete list of the 13 Archimedian solids, giving them the names by which they are still known.

What properties will the dual graph $(C_{6,8})^*$ have? (For an introduction to duality for planar graphs, see Chapter 15 of [7], for example.) Since every vertex in $C_{6,8}$ has degree 3 or 4, every region in $(C_{6,8})^*$ will be bounded by 3 or 4 edges. Further, since each region of $C_{6,8}$ has 4 bounding edges, every vertex of $(C_{6,8})^*$ will have degree 4. Finally, since $C_{6,8}$ is bipartite, every edge of $(C_{6,8})^*$ will separate a triangle from a quadrilateral, so each triangle is surrounded by 3 quadrilaterals and each quadrilateral is surrounded by 4 triangles. Drawing $(C_{6,8})^*$ on a sphere, we find $(C_{6,8})^*$ is isomorphic to the *cuboctahedron*, an Archimedian solid (see FIGURE 4). The faces of the cuboctahedron are two colorable (so that the triangles can be colored red and the squares can be blue) and it can be constructed



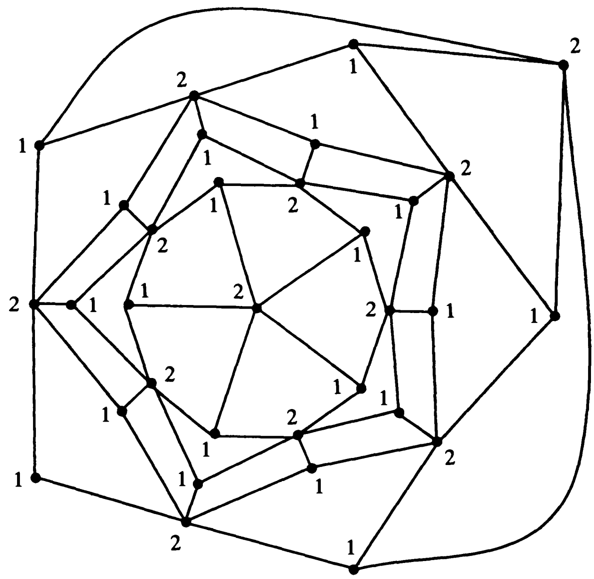
$(C_{6,8})^*$-Cuboctahedron $(D_{12,20})^*$-Icosidodecahedron

Figure 4

by truncating the eight vertices of an ordinary cube. More on building models of this solid (and many others) can be found in [4] or [6], both of which have very nice pictures.

We can also interpret $(C_{6k,8k})^*$ as the graphs of 3-dimensional solids for larger values of k. Again, each region will be bounded by either a triangle or a quadrilateral and each edge will separate a triangle from a quadrilateral. But two of the regions of $C_{6k,8k}$ are bounded by more than 4 edges, so two vertices of $(C_{6k,8k})^*$ will have degree larger than 4 (and so these solids will not be Archimedian). For example, in $(C_{12,16})^*$, there will be two vertices of degree 8 and 20 vertices of degree 4. The vertices of degree 8 (which correspond to the internal and external regions of $C_{12,16}$) will be incident to 4 triangles and 4 quadrilaterals each, while the remaining 20 vertices will have 2 triangles and 2 quadrilaterals surrounding them. The reader is encouraged to draw a picture (or even build a model) of $(C_{12,16})^*$, which is 'close to Archimedian.'

We can now apply the ideas developed above to the case where $a = 3$ and $b = 5$. Again, in the smallest possible example, the inequality in (‡) will be replaced by equality. Thus, we are looking for a planar bipartite graph in which there are 60 edges, and so $m = 20$ and $n = 12$. From Euler's formula, this graph will have 30 regions, each of which is bounded by exactly 4 edges. We can construct such a graph using some of the ideas developed above; we denote the graph $D_{12,20}$. See FIGURE 5 for a drawing of this graph.



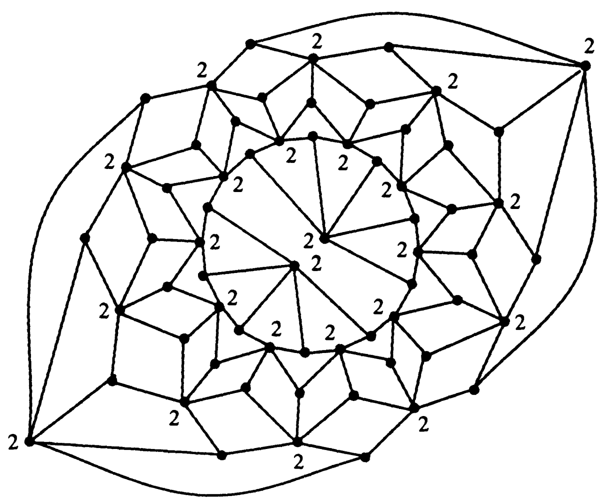
$D_{12,20}$

Figure 5

What does the dual graph look like this time? Proceeding as we did in examining $(C_{6,8})^*$, we note that every region in $(D_{12,20})^*$ will be bounded by 3 or 5 edges and every vertex of $(D_{12,20})^*$ will have degree 4. Further, every edge of $(D_{12,20})^*$ will separate a triangle from a pentagon (and so each triangle will be adjacent to 3 pentagons and each pentagon will be adjacent to five triangles).

Thus, $(D_{12,20})^*$ is also isomorphic to an Archimedian solid in which the regions (faces) are two-colorable (see FIGURE 4). This Archimedian solid is the *icosidodecahedron*, and it can be constructed by truncating the twenty vertices of a dodecahedron.

As in the $a = 3$, $b = 4$ case, we can generalize the procedure to construct an infinite family of planar bipartite graphs, denoted $D_{12k,20k}$, with $a = 3$ and $b = 5$. We modify $D_{12,20}$ in a way which is similar to the way we modified $C_{6,8}$ above to produce this family. See FIGURE 6 for a drawing of $D_{24,40}$, which has 64 vertices, 120 edges and 58 regions. (To simplify the picture, we have only labeled the 24 vertices of degree 5. All unlabeled vertices have degree 3 and should be labeled 1.) A model of the dual solid $(D_{24,40})^*$ will have 24 pentagons and 40 triangles. 56 of the 58 vertices of this solid will be incident to 2 triangles and 2 pentagons; the other 2 vertices will be incident to 4 triangles and 4 pentagons apiece. Again, this solid is 'almost' Archimedian in the same way $C_{12,16}$ is almost Archimedian.



$D_{24,40}$

Figure 6

The two Archimedian solids used here, the cuboctahedron and icosidodecahedron, are *quasi-regular* in the sense that the two kinds of faces are all regular and a face of one kind is entirely surrounded by faces of the other kind. In fact, these two solids are the only convex quasi-regular solids. (This follows from our work above, or from Chapter 2 of [2].) We also remark that the graphs $C_{6,8}$ and $D_{12,20}$ are interesting enough to have their own names: $C_{6,8}$ is the *rhombic dodecahedron* and $D_{12,20}$ is the *triacontahedron*. Models of these two graphs (on spheres, considered as solids) can be constructed so that each face is a rhombus.

These two solids were discovered by Kepler around 1611; in fact, the rhombic dodecahedron occurs in nature as a garnet crystal. See Plate I in [2] for pictures of models of these solids. Coxeter [2] gives an elegant construction to create a model of the rhombic dodecahedron. Take two solid cubes and cut one of them into 6 square pyramids, based on the six faces, all sharing the center of the cube as the common apex of the pyramids. Now glue the bases of these 6 pyramids to the 6 faces of the other cube. The resulting solid is the rhombic dodecahedron. (See also

Table 4.12 of [4].) As far as I know, the models for the higher order graphs $C_{6k,8k}$ and $D_{12k,20k}$ and their duals (for $k > 1$) have not been considered before.

Finally we remark that the question concerning which of these graphs can actually be realized as gear arrangements, which was the motivation behind these examples, remains open. Many of the graphs in this paper can easily be shown to correspond to gear arrangements.

We conclude by summarizing the results of this section.

Proposition 5. *If G is a (connected) semi-regular planar bipartite graph in which every red vertex has degree a and every blue vertex has degree b (with $a < b$), then the only possibilities for a and b are the following:*

(a) $a = 1$ *and* $b > 1$,
(b) $a = 2$ *and* $b > 2$,
(c) $a = 3$ *and* $b = 4$,
(d) $a = 3$ *and* $b = 5$.

Furthermore, each of these possibilities can be realized.

ACKNOWLEDGMENTS. I would like to thank Elizabeth McMahon for her very useful comments concerning this paper and Lorenzo Traldi for some useful discussions. I dedicate this paper to Rebecca and Hannah, whose inspiration to me is limitless.

REFERENCES

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North Holland, New York, 1976.
2. H. S. M. Coxeter, *Regular Polytopes*, (Macmillan, 1963) third edition, Dover, New York, 1973.
3. N. Hartsfield and G. Ringel, *Pearls in Graph Theory: A Comprehensive Introduction*, Academic Press, San Diego, 1990.
4. A. Pugh, *Polyhedra: A Visual Approach*, University of California Press, Berkeley, CA, 1976.
5. R. J. Trudeau, *Dots and Lines*, Kent State University Press, 1976.
6. M. J. Wenninger, *Polyhedron Models*, Cambridge University Press, New York, 1970.
7. R. J. Wilson, *Introduction to Graph Theory*, (Oliver & Boyd, 1972), third edition, Longman Scientific & Technical, Essex, England, 1985.

Dept. of Mathematics
Lafayette College
Easton, PA 18042-1781
gordong@lafayett.bitnet

A *sine qua non* for making mathematics exciting to a pupil is for the teacher to be excited about it himself; if he is not, no amount of pedagogical training will make up for the defect."

—*R. L. Wilder*