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GENERALIZED ACTIVITIES AND THE TUTTE POLYNOMIAL

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The notion of activities with respect to spanning trees in graphs was introduced by W.T. Tutte, and generalized to activities with respect to bases in matroids by H. Crapo. We present a further generalization, to activities with respect to arbitrary subsets of matroids. These generalized activities provide a unified view of several different expansions of the Tutte polynomial and the chromatic polynomial.

1. Introduction

Let E be a finite set with power set $P(E)$, and let $r = r_M : P(E) \to \mathbb{N}$ (the nonnegative integers) be the rank function of a matroid M on E. The *Tutte polynomial* (or *dichromate*) of M is a polynomial invariant that was originally introduced for graphs (cf. the account in [5]) and has been generalized to matroids [2, 3]; we will be following the account in the relevant chapter of [7]. The Tutte polynomial of a matroid M can be defined by the following subset expansion:

$$t(M; x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Alternatively, $t(M)$ can be defined recursively by the following properties: $t(\emptyset) = 1$; if e is neither an isthmus nor a loop then $t(M) = t(M-e) + t(M/e)$; if e is an isthmus then $t(M) = xt(M/e)$; and if e is a loop then $t(M) = yt(M-e)$. Here for $e \in E$ the *deletion* $M - e$ and the *contraction* M/e are both matroids on $E - \{e\}$, with rank functions obtained from r_M as follows: if $S \subseteq E - \{e\}$ then $r_{M-e}(S) = r_M(S)$ and $r_{M/e}(S) = r_M(S \cup \{e\}) - r_M(\{e\})$. Note that with these conventions, if e is an isthmus or a loop then $M - e = M/e$; although these equalities are not universally popular (cf. [6, Chapter 7]), we will use them here. The recursive description of $t(M)$ is easily deduced from the subset expansion.

A third description of the Tutte polynomial uses the notion of *basis activities*. Suppose the set E is given a (completely arbitrary) linear order, and suppose $e \in E$ and B is a basis of M. If $e \notin B$ then $B \cup \{e\}$ contains a unique circuit, and e is *externally active* or *externally inactive* with respect to B according to whether or

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not e is the least element of this circuit. If $e \in B$ then $(E - B) \cup \{e\}$ contains a unique bond, and e is *internally active* or *internally inactive* with respect to B according to whether or not it is the least element of this bond. Denoting the numbers of externally and internally active elements with respect to a basis B by $e(B)$ and $i(B)$ respectively,

$$t(M; x, y) = \sum x^{i(B)} y^{e(B)},$$

the sum taken over the set of all bases B of M.

This basis activities expansion of the Tutte polynomial is easily justified using the recursive description. For if one calculates $t(M)$ recursively, applying the recursion to the elements of E in reverse order, the result will be an expression of $t(M)$ as a sum in which each summand is obtained by contracting the elements of some basis B of M and deleting the elements of $E - B$; moreover, in the process of obtaining this summand the elements of E contracted as isthmuses will be precisely the internally active elements of B, and the elements of E deleted as loops will be precisely those that are externally active with respect to B. Thus this summand will be precisely $x^{i(B)} y^{e(B)}$.

(We should note that some authors apply the recursion to the elements of E in the given order, to justify a basis activities expansion in which the active elements are the greatest elements of certain circuits and bonds; it is not difficult to translate either approach into the other.)

The present paper had its origin in the question: can one derive the basis activities expansion of the Tutte polynomial directly from the subset expansion, without referring to the recursive description? The authors are grateful to L. H. Kauffman for pointing out that at the end of [1] R. A. Bari alluded to an affirmative answer to this question: the basis activities expansion can be obtained from the subset expansion by associating to a basis B those subsets $S \subseteq E$ that contain all of B's internally inactive elements and exclude all elements of E that are externally inactive with respect to B, and then simply grouping together those terms of the subset expansion associated to each basis. In this paper we present a generalization of Bari's answer to a theory of activities with respect to arbitrary subsets (rather than only bases) that offers a unified viewpoint on the two expansions of the Tutte polynomial given above, and many more besides.

Consider the $2^{|E|}$ different ways of 'resolving' M into an empty matroid, by deleting or contracting each element of E, in reverse order. (As mentioned above, we will refer to both deletion and contraction in a matroid by any element; for loops and isthmuses the deletion and contraction are identical.) To each such resolution we associate the set S of elements which are contracted during the resolution. We will say an element e of E is an *eventual isthmus* with respect to a subset S of E if it is deleted or contracted as an isthmus during the resolution of M associated to S; *eventual loops* are defined analogously. An element e of E is *ordinary* with respect to S if it is neither an eventual isthmus nor an eventual loop with respect to S. Each of these terms will be further modified by an adjective *internal* or *external*, according to whether or not e is in S.

More explicitly: e is an *external eventual isthmus* of S if $E - S$ contains some bond whose least element is e, and e is an *internal eventual isthmus* of S if it is in S and is an external eventual isthmus of $S - \{e\}$. Also, e is an *internal eventual loop* of S if S contains some circuit whose least element is e, and e is an *external eventual loop* of S if it is not in S and it is an internal eventual loop of $S \cup \{e\}$. We denote by $\mathrm{L}(S)$ the set of all eventual loops of S, $\mathrm{IL}(S)$ the set of internal eventual loops of S, $\mathrm{EI}(S)$ the set of external eventual isthmuses of S, and so on. Note that these definitions are interrelated through duality: $\mathrm{IL}(S)$ in M coincides with $\mathrm{EI}(E - S)$ in M^*, for instance. Various other properties of these definitions are discussed in the next section, including the following.

Theorem 1. $|\mathrm{IL}(S)| = |S| - r(S)$ *and (dually)* $|\mathrm{EI}(S)| = r(E) - r(S)$.

Theorem 2. *Suppose* $S, T \subseteq E$. *Then the following statements are equivalent*:
(i) $\mathrm{IO}(S) = \mathrm{IO}(T)$,
(ii) $\mathrm{EO}(S) = \mathrm{EO}(T)$, *and*
(iii) $\mathrm{IO}(S) \subseteq T \subseteq E - \mathrm{EO}(S)$.
Moreover, if any of these holds then $L(S) = \mathrm{L}(T)$ *and* $\mathrm{I}(S) = \mathrm{I}(T)$.

Suppose we define an equivalence relation on $P(E)$ by saying that S and T are related if $\mathrm{IO}(S) = \mathrm{IO}(T)$. Theorem 2 implies that the equivalence classes are all intervals in $P(E)$: each class has a minimal element (namely, $\mathrm{IO}(S)$ for every S in the class) and a maximal element (namely, $E - \mathrm{EO}(S)$ for every S in the class), and the class consists of those subsets of its maximal element that contain its minimal element. Moreover, by Theorem 1 each equivalence class contains a unique basis of M, which equals $\mathrm{IO}(S) \cup \mathrm{I}(S)$ for every S in the class. The idea of R. A. Bari that was mentioned above—that one should associate to a basis B all the subsets $S \subseteq E$ that contain all of B's internally inactive elements and contain no elements that are externally inactive with respect to B—is essentially that one should associate to B all the elements of its equivalence class.

We say that a subset $S \subseteq E$ is of *type* (s, t, u, v) if $|\mathrm{IL}(S)| = s$, $|\mathrm{EL}(S)| = t$, $|\mathrm{II}(S)| = u$, and $|\mathrm{EI}(S)| = v$; also, we denote by $n(s, t, u, v)$ the number of subsets of type (s, t, u, v). The following consequence of Theorem 2 shows that these numbers $n(s, t, u, v)$ are highly interdependent.

Corollary 1.

$$n(s, t, u, v) = \binom{s+t}{s}\binom{u+v}{u} n(0, s+t, 0, u+v).$$

Let R be any commutative ring with unity that contains the polynomial ring $\mathbb{Z}[x, y]$. A *coefficient function* for the Tutte polynomial is any function

$c: P(E) \times E \to R$ with the property that $c(S, e)$ depends only on those elements of S that are preceded by e in the ordering of E. In the third section we will prove

Theorem 3. *For any such coefficient function,*

$$t(M; x, y) = \sum_{S \subseteq E} \Big(\prod_{\mathrm{EI}} c(S, e)\Big)\Big(\prod_{\mathrm{II}} (x - c(S, e))\Big)\Big(\prod_{\mathrm{EL}} c(S, e)\Big)\Big(\prod_{\mathrm{IL}} (y - c(S, e))\Big),$$

where $\mathrm{EI} = \mathrm{EI}(S)$ *and so on.*

We will also see that appropriate choices of the coefficient function in Theorem 3 yield the subset expansion of $t(M)$, the basis activities expansion, and also expansions indexed by the spanning sets of M and the independent sets of M. The independent sets expansion is of particular interest, as it generalizes the 'broken circuit' expansion of the chromatic polynomial of a graph, which was introduced nearly sixty years ago by Whitney [8].

2. The generalized activities

An obvious consequence of the definitions of the first section is

Proposition 2.1. *Suppose* $S \subseteq T \subseteq E$. *Then* $\mathrm{L}(S) \subseteq \mathrm{L}(T)$ *and* $\mathrm{I}(S) \supseteq \mathrm{I}(T)$.

Another simple property is

Proposition 2.2. *If* $S \subseteq E$ *then* $\{\mathrm{IL}(S), \mathrm{EL}(S), \mathrm{II}(S), \mathrm{EI}(S), \mathrm{IO}(S), \mathrm{EO}(S)\}$ *is a partition of* E.

Proof. The only aspect of this that is not immediately obvious is the assertion that $\mathrm{L}(S)$ and $\mathrm{I}(S)$ are disjoint. This assertion is a simple consequence of the fact that a bond and a circuit cannot have only a single element in common. □

It will be convenient to use the notation $E = \{e_1, \ldots, e_m\}$, with indices reflecting the order of E; if $i \leq j$ we will use $[e_i, e_j]$ to denote $\{e_i, \ldots, e_j\}$.

We only prove the first statement of Theorem 1; the other follows from duality. Choose $A \subseteq S$ to be the maximal independent set which is last lexicographically. We show that $\mathrm{IL}(S) = S - A$.

If $e_k \in S - A$ then $(A \cap [e_{k+1}, e_m]) \cup \{e_k\}$ is dependent, as otherwise it would be contained in some maximal independent set lexicographically preceded by A. Thus e_k is the least element of some circuit contained in S, so $e_k \in \mathrm{IL}(S)$; this shows that $S - A \subseteq \mathrm{IL}(S)$. On the other hand, if $e_k \in \mathrm{IL}(S) \cap A$, let $C \subseteq S$ be a circuit in which e_k is the least element. Then we could replace e_k in A with some other member of C, contradicting the choice of A. Hence $\mathrm{IL}(S) \subseteq S - A$.

As $r(S) = |A|$, this suffices to prove Theorem 1.

Let $S \subseteq E$. Define a sequence of matroids $M_k(S)$, $1 \leq k \leq m$, by $M_m(S) = M$ and for $k < m$,

$$M_k(S) = (M - ((E - S) \cap [e_{k+1}, e_m])/(S \cap [e_{k+1}, e_m]).$$

That is, $M_k(S)$ is the matroid obtained from M by contracting or deleting each $e \in [e_{k+1}, e_m]$, according to whether or not $e \in S$. The following lemma is a direct consequence of this definition, and the fact that the contraction and deletion of an isthmus or loop coincide.

Lemma 2.3. *For* $1 \leq k \leq m$, $M_k(S) = M_k(\mathrm{IO}(S)) = M_k(E - \mathrm{EO}(S))$.

Lemma 2.4. *Let* $S, T \subseteq E$. *If* $M_k(S) = M_k(T)$ *for every* k *then* $\mathrm{I}(S) = \mathrm{I}(T)$, $\mathrm{L}(S) = \mathrm{L}(T)$, *and* $\mathrm{O}(S) = \mathrm{O}(T)$.

Proof. Note that $e_k \in \mathrm{I}(S)$ if and only if e_k is an isthmus in $M_k(S)$, and similarly $e_k \in \mathrm{I}(T)$ if and only if it is an isthmus in $M_k(T)$. Since $M_k(S) = M_k(T)$ for every k, it follows that $\mathrm{I}(S) = \mathrm{I}(T)$. Similarly, $\mathrm{L}(S) = \mathrm{L}(T)$. Finally, $\mathrm{O}(S) = \mathrm{O}(T)$ follows from Proposition 2.2. □

Corollary 2.5. *Suppose* $e_k \in \mathrm{I}(S) \cup \mathrm{L}(S)$. *Then*

$$\mathrm{I}(S - \{e_k\}) = \mathrm{I}(S) = \mathrm{I}(S \cup \{e_k\}), \qquad \mathrm{L}(S - \{e_k\}) = \mathrm{L}(S) = \mathrm{L}(S \cup \{e_k\}),$$
$$\mathrm{IO}(S - \{e_k\}) = \mathrm{IO}(S) = \mathrm{IO}(S \cup \{e_k\}), \quad \text{and}$$
$$\mathrm{EO}(S - \{e_k\}) = \mathrm{EO}(S) = \mathrm{EO}(S \cup \{e_k\}).$$

Proof. Obviously $M_j(S - \{e_k\}) = M_j(S) = M_j(S \cup \{e_k\})$ for every j. The corollary now follows from Lemma 2.4. □

We turn now to the proof of Theorem 2. Suppose first that $\mathrm{IO}(S) \subseteq T \subseteq E - \mathrm{EO}(S)$. Then by applying Corollary 2.5 repeatedly if necessary, we can delete $S - T$ from S one element at a time, and adjoin $T - S$ to the result, thus transforming S into T. Each such deletion or adjunction involves an element of $\mathrm{I}(S) \cup \mathrm{L}(S)$, so Corollary 2.5 gives $\mathrm{IO}(S) = \mathrm{IO}(T)$.

Now suppose $\mathrm{IO}(S) = \mathrm{IO}(T)$. By Lemma 2.3, $M_k(T) = M_k(\mathrm{IO}(T)) = M_k(\mathrm{IO}(S)) = M_k(S)$ for $1 \leq k \leq m$. By Lemma 2.4, $\mathrm{O}(S) = \mathrm{O}(T)$, so $\mathrm{EO}(S) = \mathrm{O}(S) - \mathrm{IO}(S) = \mathrm{O}(T) - \mathrm{IO}(T) = \mathrm{EO}(T)$.

If $\mathrm{EO}(S) = \mathrm{EO}(T)$, then Lemma 2.3 gives $M_k(T) = M_k(E - \mathrm{EO}(T)) = M_k(E - \mathrm{EO}(S)) = M_k(S)$ for every k. By Lemma 2.4, it follows that $\mathrm{O}(S) = \mathrm{O}(T)$. Therefore, $\mathrm{IO}(S) = \mathrm{O}(S) - \mathrm{EO}(S) = \mathrm{O}(T) - \mathrm{EO}(T) = \mathrm{IO}(T) \subseteq T \subseteq E - \mathrm{EO}(T) = E - \mathrm{EO}(S)$. This completes the proof of Theorem 2. □

To prove Corollary 1, let X be the set of all $S \subseteq E$ of type (s, t, u, v), and let Y be the set of all triples (T, T', T'') such that T is of type $(0, s + t, 0, u + v)$, T'

consists of s elements of L(T), and T'' consists of u elements of I(T). Consider the function $f: X \to Y$ given by $f(S) = (\mathrm{IO}(S), \mathrm{IL}(S), \mathrm{II}(S))$. Clearly f is injective, and Theorem 2 implies that f is surjective too; for if $(T, T', T'') \in Y$ then $(T, T', T'') = f(T \cup T' \cup T'')$. Thus

$$n(s, t, u, v) = |X| = |Y| = n(0, s+t, 0, u+v)\binom{s+t}{s}\binom{u+v}{v}.$$

It is worth noting that the various integers $n(s, t, u, v)$ are independent of the choice of the ordering of E used to define the generalized activities, though of course the types of particular subsets are not. To see why this is so, observe first that Corollary 1 implies that $n(s, t, u, v) = n(s, t, v, u)$. Next, observe that by Theorem 1 $n(0, s+t, 0, u+v) = n(0, s+t, u+v, 0)$ is the number of bases B with $i(B) = u + v$ and $e(B) = s + t$; a fundamental consequence of the existence of the basis activities expansion of $t(M)$ is that this number is independent of the choice of the ordering of E. It follows that $n(s, t, u, v)$ is independent of the choice of ordering.

3. Expansions of the Tutte polynomial

The proof of Theorem 3 is quite similar to the justification of the basis activities expansion given in the introduction. Consider the process of calculating $t(M)$ recursively, applying the following recursion to the elements of E (in reverse order). If at a certain stage in the calculation we are applying the recursion to an element $e = e_k$ of a matroid $M' = M_k(S)$, where S consists of those elements of E that have been contracted in arriving at this stage, then:

$$t(M') = t(M' - e) + t(M'/e)$$

if e is neither a loop nor an isthmus in M';

$$t(M') = c(S, e)t(M' - e) + (x - c(S, e))t(M'/e)$$

if e is an isthmus in M'; and

$$t(M') = c(S, e)t(M' - e) + (y - c(S, e))t(M'/e)$$

if e is a loop in M'. The final result of the calculation is to express $t(M)$ as a sum of $2^{|E|}$ terms; the term obtained by contracting the elements of a subset $S \subseteq E$, and deleting the elements of $E - S$, will be precisely

$$\Big(\prod_{\mathrm{EI}(S)} c(S, e)\Big)\Big(\prod_{\mathrm{II}(S)} (x - c(S, e))\Big)\Big(\prod_{\mathrm{EL}(S)} c(S, e)\Big)\Big(\prod_{\mathrm{IL}(S)} (y - c(S, e))\Big).$$

This completes the proof of Theorem 3. □

We now present several families of examples of coefficient functions, and the resulting expansions of the Tutte polynomial. All the coefficient functions we consider have the property that $c(S, e)$ depends only on whether e is in $\mathrm{O}(S)$, $\mathrm{L}(S)$, or $\mathrm{I}(S)$. Clearly any function with this property is a coefficient function; moreover, the resulting expansion of the Tutte polynomial is independent of the value of $c(S, e)$ for $e \in \mathrm{O}(S)$, so we will not bother to specify this value.

Example 3.1. Let $c(S, e) = 1$ for $e \in \mathrm{L}(S)$, and $c(S, e) = x - 1$ for $e \in \mathrm{I}(S)$. Theorem 3 gives

$$t(M) = \sum_{S \subseteq E} (x-1)^{|\mathrm{EI}(S)|}(y-1)^{|\mathrm{IL}(S)|}.$$

By Theorem 1, this is simply the subset expansion of $t(M)$. Three related examples are obtained by replacing $c(S, e)$ by $y - 1$ for $e \in \mathrm{L}(S)$, or by 1 for $e \in \mathrm{I}(S)$, or both. The resulting expansions are

$$\begin{aligned} t(M) &= \sum (x-1)^{|\mathrm{EI}(S)|}(y-1)^{|\mathrm{EL}(S)|} \\ &= \sum (x-1)^{|\mathrm{II}(S)|}(y-1)^{|\mathrm{IL}(S)|} \\ &= \sum (x-1)^{|\mathrm{II}(S)|}(y-1)^{|\mathrm{EL}(S)|}. \end{aligned}$$

Theorem 2 implies that these four expansions are all identical. For instance, consider the function $f : P(E) \to P(E)$ given by $f(S) = \mathrm{IO}(S) \cup \mathrm{EL}(S) \cup \mathrm{EI}(S)$. It is a bijection with the property that S is of type (s, t, u, v) if, and only if, $f(S)$ is of type (t, s, v, u); it can be used to transform the first of the four expansions (the subset expansion) into the last.

Example 3.2. Let $c(S, e) = y$ for $e \in \mathrm{L}(S)$ and $c(S, e) = 0$ for $e \in \mathrm{I}(S)$. Using this coefficient function in Theorem 3 yields the basis activities expansion of the Tutte polynomial. Other examples in this family are obtained by redefining $c(S, e)$ to be 0 for $e \in \mathrm{L}(S)$, or x for $e \in \mathrm{I}(S)$, or both; once again, the resulting expansions of $t(M)$ differ only cosmetically from the basis activities expansion and from each other.

Example 3.3. Another coefficient function that yields the basis activities expansion is given by $c(S, e) = y/2$ for $e \in \mathrm{L}(S)$ and $c(S, e) = x/2$ for $e \in \mathrm{I}(S)$. It gives rise to the expansion

$$t(M) = \sum_{S \subseteq E} x^{|\mathrm{I}(S)|} y^{|\mathrm{L}(S)|} 2^{-|\mathrm{I}(S)| - |\mathrm{L}(S)|}.$$

By Theorem 2, for each $S \subseteq E$ there are $2^{|\mathrm{I}(S)| + |\mathrm{L}(S)|}$ subsets T with $\mathrm{IO}(S) = \mathrm{IO}(T)$, and exactly one of these is a basis. It follows that by grouping together the terms of this expansion that arise from subsets with the same internally ordinary elements we can arrive at the basis activities expansion of $t(M)$, just as we can by grouping together terms in the subset expansion.

Example 3.4. Another family of examples starts with the coefficient function given by $c(S, e) = 0$ for $e \in \mathrm{I}(S)$ and $c(S, e) = 1$ for $e \in \mathrm{L}(S)$. The resulting expansion is

$$t(M) = \sum_{\mathrm{EI}(S)=\emptyset} x^{|\mathrm{II}(S)|}(y-1)^{|\mathrm{IL}(S)|};$$

note that this sum is indexed by the spanning sets of M. This coefficient function can be modified, replacing $c(S, e)$ by x for $e \in \mathrm{I}(S)$, or by $y - 1$ for $e \in \mathrm{L}(S)$, or both; as before, the resulting expansions of $t(M)$ differ only in appearance from the one just given.

Example 3.5. Our final family of examples starts with the coefficient function given by $c(S, e) = x - 1$ for $e \in \mathrm{I}(S)$ and $c(S, e) = y$ for $e \in \mathrm{L}(S)$. Theorem 3 yields the expansion

$$t(M) = \sum_{\mathrm{IL}(S)=\emptyset} (x-1)^{|\mathrm{EI}(S)|} y^{|\mathrm{EL}(S)|},$$

indexed by the independent sets of M. Replacing $c(S, e)$ by 1 for $e \in \mathrm{L}(S)$, or by 0 for $e \in \mathrm{L}(S)$, or both, produces other coefficient functions in this family; as in the earlier examples, the resulting expansions of $t(M)$ are not different from this one in any essential way.

This independent sets expansion of $t(M)$ is of special interest because it specializes to a well-known expansion of the chromatic polynomial of a graph. Recall that if G is a graph with n vertices, k connected components, and associated circuit matroid M, then the chromatic polynomial of G is

$$P(G; \lambda) = (-\lambda)^k(-1)^n t(M; 1-\lambda, 0).$$

Noting that an independent $S \subseteq E = E(G)$ has $r(S) = |S|$ and hence $|\mathrm{EI}(S)| = r(E) - |S| = n - k - |S|$, the independent sets expansion of $t(M)$ yields

$$P(G; \lambda) = \sum_{L(S)=\emptyset} (-\lambda)^{n-|S|}(-1)^n = \sum_{L(S)=\emptyset} (-1)^{|S|}\lambda^{n-|S|}.$$

This expansion of the chromatic polynomial is due to Whitney [8].

Another expansion of the chromatic polynomial, given by R. A. Bari in Theorem 4.1 of [1], may be obtained similarly, by specializing the basis activities expansion of $t(M)$.

4. Weighted matroids

A *weighted matroid* is a matroid M together with a function $w: E \to R$, where R is some commutative ring with unity. In [4] a generalization of the dichromatic polynomial (an invariant of graphs that is closely related to the Tutte polynomial, cf. [5]) to weighted graphs was presented; the corresponding generalization of the Tutte polynomial to weighted matroids is given by the subset expansion

$$t(M; x, y) = \sum_{S \subseteq E} \Big(\prod_{e \in S} w(e)\Big)(x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Clearly the Tutte polynomial of an unweighted matroid can be recovered from this one by assigning all elements the weight 1. Also, note that if $Z \subseteq E$ consists of the elements of weight zero then $t(M)(x-1)^{r(E-Z)-r(E)} = t(M-Z)$, and consequently $t(M)(x-1)^{r(E-Z)-r(E)}$ is unchanged if elements of weight 0 are adjoined to, or deleted from, M.

This polynomial has a recursive description:

$$t(\emptyset) = 1; \qquad t(M) = t(M-e) + w(e)t(M/e)$$

if e is neither an isthmus nor a loop;

$$t(M) = (x + w(e) - 1)t(M/e)$$

if e is an isthmus; and

$$t(M) = (yw(e) - w(e) + 1)t(M-e)$$

if e is a loop.

We leave it to the reader to prove the following generalization of Theorem 3, and to generalize the examples of Section 3 to this context.

Theorem 4. *Let c be a coefficient function mapping $P(E) \times E$ into some commutative ring with unity containing $R[x, y]$. Then*

$$t(M; x, y) = \sum_{S \subseteq E} \Big(\prod_{\mathrm{EI} \cup \mathrm{EL}} c(S, e)\Big)\Big(\prod_{\mathrm{II}} (x + w(e) - c(S, e) - 1\Big) \times \Big(\prod_{\mathrm{IL}} (yw(e) - w(e) - c(S, e) + 1)\Big),$$

where $\mathrm{EI} = \mathrm{EI}(S)$ *and so on.*

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