ELSEVIER Discrete Mathematics 147 (1995) 297–300

DISCRETE MATHEMATICS

Note

Trees with the same degree sequence and path numbers[☆]

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Received 27 September 1993; revised 14 June 1994

Abstract

We give an elementary procedure based on simple generating functions for constructing n (for any $n \geqslant 2$) pairwise non-isomorphic trees, all of which have the same degree sequence and the same number of paths of length k for all $k \geqslant 1$. The construction can also be used to give a sufficient condition for isomorphism of caterpillars.

In [2], a 2-variable polynomial that is closely related to the familiar Tutte polynomial of a graph or matroid is introduced and considered for trees. Two tree invariants are of special interest here. In particular, it is shown that for a given tree T, the polynomial $f(T; t, z)$ determines the degree sequence of T as well as the number of paths of length k for all values of $k \geqslant 1$. Thus, if $f(T_1) = f(T_2)$, then the trees T_1 and T_2 must share the same degree sequence and the same number of paths of length k for all k (see Proposition 18 of [2]). In this context, it is natural to ask whether these two invariants uniquely determine the tree. We answer this question in the negative here, giving a procedure for constructing an infinite family of pairs of non-isomorphic trees, each pair of which has the same degree sequence and the same number of paths of length k for all $k \geqslant 1$. In fact, the construction can be used to create an arbitrarily large family of trees, all of which share the same degree sequence and same path numbers. This construction is completely elementary and also gives a sufficient condition for isomorphism of caterpillars.

We assume the reader is familiar with graph theory; a standard reference is [1]. A *caterpillar* is a tree for which the set of vertices that are not leaves forms a path, called the spine. (See Fig. 1 for an example.) Let T be a caterpillar and let $u_0, \ldots, u_n$

☆ This work was supported by an NSF REU grant at Lafayette College.
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SSDI 0012-365X(94)00278-9

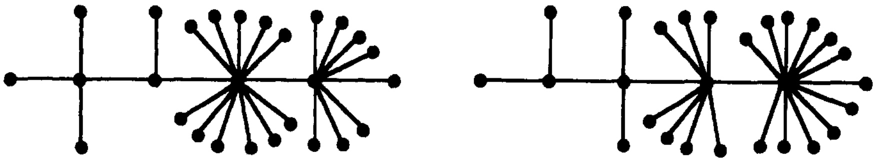

Fig. 1

denote the vertices along its spine, where u_i is adjacent to u_{i+1} for $0 \leqslant i < n$. Then define $g(T;x)$ to be the following generating function associated with T;

$$g(T;x) = e_0 + e_1x + e_2x^2 + \cdots + e_nx^n$$

where $e_i = \deg(u_i) - 1$ for $0 \leqslant i \leqslant n$. Evidently, there are two possible generating functions which can be associated to T depending on whether the sequence $e_0, e_1, \ldots, e_n$ or its reverse $e_n, e_{n-1}, \ldots, e_0$ is used. To distinguish between these two choices, choose the lexicographically smaller of the two possible sequences to define $g(T;x)$. The generating function $g(T;x)$ obviously determines the caterpillar uniquely.

For a caterpillar T with e_i defined as above, let P_k denote the number of paths of length $k, 1 \leqslant k \leqslant n+2$. It is easy to determine the P_k in terms of the e_i.

Lemma 1. *Let T be a caterpillar with P_k paths of length k for $3 \leqslant k \leqslant n+2$ and with e_i sequence defined above. Then $P_k = \sum_{i=0}^{n-k+2} e_ie_{i+k-2}$. Furthermore, $P_1 = 1 + \sum_{i=0}^{n} e_i$ and*

$$P_2 = \sum_{i=0}^{n} \binom{e_i + 1}{2}.$$

Proof. The result is trivial for paths of length 1 and 2. For paths of length k for $k \geqslant 3$, note that such a path can only be formed by selecting two vertices u_i and u_{i+k-2} along the spine of T to be the second and penultimate vertices (respectively) of the path. The number of paths which use u_i and u_{i+k-2} in this way is just e_ie_{i+k-2}, since there are e_i choices for the initial vertex of the path and e_{i+k-2} choices for the terminal vertex. Summing over all such choices gives the formula. □

We can now make a connection between the path numbers P_k and the generating function $g(T;x)$. For a polynomial $p(x)$ with integer coefficients, let $R(p(x))$ be the polynomial obtained from $p(x)$ by reversing the coefficient sequence. Thus, if the degree of $p(x)$ is n, then $R(p(x)) = x^np(x^{-1})$. We omit the straightforward proof of the next lemma.

Lemma 2. *Let T be a caterpillar and let $h(T;x) := g(T;x)R(g(T;x))$. Then*

$$h(T;x) = P_{n+2} + P_{n+1}x + P_nx^2 + \cdots + P_3x^{n-1} + \sum e_i^2x^n + P_3x^{n+1} + P_4x^{n+2} + \cdots + P_{n+2}x^{2n}.$$

We now show how to use the polynomial $h(T;x)$ to produce non-isomorphic caterpillars with the same degree sequence and path numbers.

Example 3. Let T_1 be a caterpillar with

$$g(T_1;x) = (2x+3)(4x^2+1) = 8x^3 + 12x^2 + 2x + 3.$$

Then

$$h(T_1;x) = (8x^3 + 12x^2 + 2x + 3)(3x^3 + 2x^2 + 12x + .8)$$
$$= (2x+3)(4x^2+1)(3x+2)(x^2+4)$$

Thus, if we let T_2 be the caterpillar with

$$g(T_2;x) = (3x+2)(4x^2+1) = 12x^3 + 8x^2 + 3x + 2,$$

then

$$h(T_2;x) = (12x^3 + 8x^2 + 3x + 2)(2x^3 + 3x^2 + 8x + 12)$$
$$= (3x+2)(4x^2+1)(2x+3)(x^2+4)$$
$$= h(T_1;x)$$

Hence T_1 and T_2 have the same degree sequence and the same number of paths of length k for all $k \geqslant 1$. (Obviously two caterpillars with the same unordered e_i sequence as defined above will have the same degree sequence. See Fig. 1.)

The technique used in Example 3 can be used to generate infinitely many such pairs of caterpillars. We formalize this in the next theorem.

Theorem 4. *Let T_1 be a caterpillar with*

$$g(T_1;x) = \prod_{i=0}^{m} (a_i x^{2^i} + b_i)$$

for positive integers a_i and b_i for $0 \leqslant i \leqslant m$. Let T_2 be a caterpillar with $g(T_2;x)$ formed by reversing some of the factors of $g(T_2;x)$. Then T_1 and T_2 have the same degree sequence and the same number of paths of length k for all $k \geqslant 1$.

Proof. The uniqueness of the binary representation of the integers insures that each coefficient in $g(T_1;x)$ appears as a coefficient in $g(T_2;x)$, so T_1 and T_2 have the same (unordered) e_i sequence, hence they have the same degree sequence. This also implies (by Lemma 1) that they have the same number of paths of length 1 and 2. To show that T_1 and T_2 have the same number of paths of length $k \geqslant 3$, note that $h(T_1;x) = h(T_2;x)$ since the reversing operation has the multiplicative homomorphism property in the polynomial ring $\boldsymbol{Z}[x]$ (so $R(p(x)q(x)) = R(p(x))R(q(x))$) and factorization in $\boldsymbol{Z}[x]$ is unique. Thus, by Lemmas 1 and 2, we see that T_1 and T_2 have the same number of paths of length k for all $k \geqslant 3$. This completes the proof. □

Corollary 5. *For any positive integer M, there are M pairwise non-isomorphic caterpillars, all sharing the same degree sequence and the same number of paths of length k for all* $k \geqslant 1$.

Proof. Let T_1 be a caterpillar with $g(T_1; x) = \prod_{i=0}^{m} (a_i x^{2^i} + b_i)$, as in Theorem 4, with $a_i \neq b_i$ for all i. Of the 2^{m+1} possible polynomials $g(T_2; x)$ formed by reversing some of the factors of $g(T_1; x)$, note that every such caterpillar T_2 appears twice, since reversing a set of factors and reversing the complement of that set produce isomorphic caterpillars. Since $a_i \neq b_i$ for all i, uniqueness of binary representation implies the collection of 2^m caterpillars are pairwise non-isomorphic. □

We can also use these ideas to produce a sufficient condition for two caterpillars to be isomorphic.

Corollary 6. *Let T_1 and T_2 be two caterpillars with the same degree sequence and the same number of paths of length k for all $k \geqslant 1$. If $g(T_1; x)$ is irreducible over $\mathbf{Z}[x]$, then T_1 and T_2 are isomorphic.*

Proof. By Lemma 2, we have $h(T_1; x) = h(T_2; x)$. By the homomorphism property of $R(p(x))$, $g(T_1; x)$ is irreducible iff $R(g(T_1; x))$ is irreducible. Thus $g(T_2; x) = g(T_1; x)$, so T_1 and T_2 are isomorphic. □

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, Amsterdam, 1976).
[2] S. Chaudhary and G. Gordon, Tutte polynomials for trees, J. Graph Th. 15 (1991) 317–331.