JOURNAL OF COMBINATORIAL THEORY, Series B 59, 132–155 (1993)

A Tutte Polynomial for Partially Ordered Sets

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Received September 25, 1990

DEDICATED TO G. G.

We investigate the Tutte polynomial of a greedoid associated to a partially ordered set. In this case, we explore the deletion-contraction formula in two ways and develop an antichain expansion for the polynomial. We show that the polynomial can determine the number of orders ideals, order filters and antichains of all sizes of a poset P, but neither the number of chains, multichains, extensions, nor the dimension of P. We show how to compute the polynomial for the direct sum, ordinal sum, and ordinal product of two posets, but show that this cannot be done for a direct product. We also show it is possible to determine $f(P^*)$ from $f(P)$, where P^* is the dual poset of P. We also consider the idea of *feasible isomorphism* of two greedoids and show that two feasibly isomorphic greedoids have the same polynomial. © 1993 Academic Press, Inc.

INTRODUCTION

The Tutte polynomial is a two-variable polynomial which has been defined and studied in depth for graphs and matroids. Extensive introductions to the theory can be found in [Br-O, Br]. Recently, this definition has been extended to greedoids [G-M], generalizing the one-variable greedoid polynomial described in [Bj-Z]. In this paper, we concentrate on applying the two-variable polynomial to certain families of greedoids, especially antimatroids and partially ordered sets (posets).

The idea of using a "Tutte-like" polynomial for posets is not new. The zeta polynomial, order polynomial, strict-order polynomial, and rank generating function have all received much attention. Furthermore, the Möbius function of a poset is extremely important and is related to many of these polynomial invariants. As another example, in problem 26 on page 158 of Stanley's remarkable book [S], a two-variable polynomial is defined on posets and two properties of the polynomial are derived. (This polynomial is related to the polynomial we will consider here—see Example 3.5.)

A different approach can be taken through the context of greedoids. Since posets are an example of greedoids with the order ideals playing the

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role of the feasible sets, any Tutte polynomial defined on greedoids is a poset polynomial. However, the *greedoid polynomial* $\lambda_G(y)$ described in [Bj-Z] is a one-variable polynomial which gives no information about posets since $\lambda_{G(P)}(y) \equiv 1$ for any poset P, where $G(P)$ is the greedoid associated with the poset P. Using the corank-nullity formulation (instead of a basis activities formulation) of the Tutte polynomial leads to a two-variable polynomial $f(G)$ [G-M] which is a generalization of $\lambda_G(y)$ and has many desirable properties (e.g., a deletion–contraction recurrence and a direct sum decomposition). For example, in Theorem 2.8 of [G-M], it is shown that when this two-variable polynomial is restricted to rooted trees (which are also a class of greedoids), the polynomial completely determines the rooted tree. (See Theorem 3.10 for a generalization.)

The point of our study is to investigate what properties hold for this polynomial and how these properties reflect various aspects of the poset structure, rather than to solve poset enumeration problems which could not be solved by other means. Furthermore, we do not expect the polynomial to be an effective computational tool in general (since all known methods of computing the polynomial have exponential complexity), but rather to be of interest from a theoretical viewpoint. For example, the equivalence of various expansions may lead to a new way of thinking about an existing problem. Example 2.7 and the applications of Section 3 are attempts to give specific instances of how such an approach could be used.

In Section 2, we concentrate on posets, developing two different deletion–contraction recursions for the polynomial and an expansion in terms of antichains. We point out that contraction is only defined for feasible sets and, in particular, contraction of a singleton is only valid when the singleton is feasible, i.e., when the singleton is minimal in the poset. (One recursion (Proposition 2.4) works on minimal elements, the other (Proposition 2.5) works on maximal elements.) The antichain expansion is a consequence of a feasible set expansion (Theorem 2.2) which is valid for antimatroids. This expansion will be central for the remainder of this paper. We also discuss an *activities* expansion, based on [G-T], which gives all the other expansions as various special cases.

In Section 3, we concentrate on examples, counterexamples, applications, and evaluations. Example 3.1 gives the minimal example of nonisomorphic posets P and Q with $f(P) = f(Q)$. This example shows that the polynomial cannot determine any information about any invariant in which P and Q differ. Loosely speaking, $f(P)$ gives substantial information about antichains (which are in one-to-one correspondence with order ideals), but little or no information about chains. Three applications we discuss are the equivalence of $f(P^*)$ and the two-variable polynomial defined in [S] (Example 3.5), the fact that the dimension of P cannot be determined from $f(P)$ (Example 3.7), and the construction of a class of posets which can be uniquely reconstructed from their polynomials (Theorem 3.10).

Section 4 is primarily concerned with poset operations. We discuss the behavior of the polynomial under direct sum, ordinal sum, and ordinal product, obtaining formulas for the polynomial in each of these cases. The counterexample of 4.2 shows no such formula is possible for the direct product, however. The main theorem (Theorem 4.5) proves that if $f(P) = f(Q)$, then $f(P^*) = f(Q^*)$. The proof of 4.5 uses the fundamental theorem of finite distributive lattices which gives a bijection between finite posets P and finite distributive lattices $J(P)$. A discussion of this theorem, which is credited to Birkhoff, can be found in [S], for example. Despite the close connection between P and $J(P)$, we show in Example 4.6 that $f(P) = f(Q)$ with $f(J(P)) \neq f(J(Q))$ is possible.

Section 5 introduces the idea of a *feasible isomorphism* between greedoids. Two greedoids G and H of cardinality n and rank r will be called *feasibly isomorphic* if, for all $0 \leqslant k \leqslant r$, there is a permutation π_k of $\{1, 2, ..., n\}$ which induces a bijective map from the set of feasible sets of size k in G to the set of feasible sets of size k in H. For matroids, feasible isomorphism and matroid isomorphism are the same. For arbitrary greedoids, we show (Theorem 5.2 and Example 5.4) that this equivalence lies properly between equality of polynomials and greedoid isomorphism. In other words, if G and H are greedoids, then $G \cong H \Rightarrow G$ and H are feasibly isomorphic $\Rightarrow f(G) = f(H)$, but neither implication is reversible.

I thank Professors Tom Brylawski, Elizabeth McMahon, and Lorenzo Traldi for many stimulating discussions and the anonymous referee for an extremely detailed report which substantially improved the final version of this paper.

1. Mathematical Preliminaries

We now give some of the basic definitions we will require: a standard reference for the graph theory is [T]; for the matroid theory see [We] or [Wh]; for the greedoid theory see [Bj-Z]; and the poset theory can be found in Chapter 3 of [S], for example. We will assume the reader is familiar with the basic terms from poset theory. We also remark that much of the underlying theory of greedoids is based on a series of papers by Korte and Lovasz. See [K-L1, K-L2, or K-L3] for a sample. A short survey of matroids and antimatroids appears in [Di].

Matroids and greedoids can be defined in many equivalent ways. Following [Bj-Z], let E be a finite set and let $\mathscr{F}$ be a non-empty set of subsets of E. Then $\mathscr{F}$ is called a *set system.* A *greedoid* G is a pair $G = (E, \mathscr{F})$, where the set system $\mathscr{F}$ satisfies both G1 and G2:

G1. $\emptyset \neq X \in \mathscr{F} \Rightarrow X - \{x\} \in \mathscr{F}$ for some $x \in X$.

G2. $X, Y \in \mathscr{F}$ and $|X| < |Y| \Rightarrow X \cup \{x\} \in \mathscr{F}$ for some $x \in Y - X$.

The subsets in $\mathscr{F}$ are called the *feasible* sets of the greedoid. Note that since E is finite and $\mathscr{F}$ is non-empty, G1 implies $\emptyset \in \mathscr{F}$. Axiom G1 is called *accessibility* and can be interpreted as saying every feasible set can be built up from $\emptyset$ by repeated addition of singletons, each of which preserves feasibility. A strengthening of G1 is the axiom M1:

M1. $X \subseteq Y \in \mathscr{F} \Rightarrow X \in \mathscr{F}$.

A set system satisfying M1 is termed *hereditary* or a *simplicial complex*. A pair $G = (E, \mathscr{F})$ is a *matroid* if the set system $\mathscr{F}$ satisfies both M1 and G2. Every matroid is a greedoid, but there are many interesting examples of greedoids which are not matroids. Maximal feasible sets are called *bases* of the greedoid, and it is an easy but important fact that all bases in a greedoid G have the same cardinality.

Let $G = (E, \mathscr{F})$ be a greedoid. For $A \subseteq E$, define the *rank of* A, denoted $r(A)$, to be the size of a largest feasible subset of A. Thus, $r(A) = \max\{|F| : F \subseteq A$ and F is feasible$\}$. (We define the *rank of the greedoid* G by $r(G) = r(E)$.) Thus, rank can be regarded as a function from the power set of E to the non-negative integers. It is possible to define greedoids in terms of this rank function, but we will not need to take this approach here. (We also note that, for matroids, this definition of rank coincides with the classical one.)

The *Tutte polynomial*, or, more precisely, the *corank-nullity polynomial* of a greedoid G on the ground set E with rank function r is defined by

T1. $f(G; t, z) = \sum_{A \subseteq E} t^{r(E) - r(A)} z^{|A| - r(A)}$.

This polynomial generalizes the Tutte polynomial of graphs and matroids and has many interesting evaluations. For example, $f(G; 0, 0) =$ the number of bases of G; $f(G; 1, 0) =$ the number of feasible sets; $f(G; 0, 1) =$ the number of *spanning* sets (i.e., the number of sets which contain a basis); and $f(G; 1, 1) = 2^{|E|}$. See [Br-O or Br] for more information about the Tutte polynomial of matroids.

Let G be a greedoid on the ground set E with feasible sets $\mathscr{F}$. We now define *deletion* of an arbitrary subset A and *contraction* by a feasible set A by specifying the feasible sets of the deletion $G - A$ and the contraction G/A:

$G - A$ is a greedoid on $E - A$ with feasible sets $\{F \subseteq E - A : F \in \mathscr{F}\}$

G/A is a greedoid on $E - A$ with feasible sets $\{F \subseteq E - A : F \cup A \in \mathscr{F}\}$.

If A is not feasible, then the contraction G/A will not be a greedoid (as $\emptyset$

will not be feasible). Although there are other approaches to this difficulty, we will always contract by a feasible set.

The recursive description we give now is proven in Proposition 3.2 of [G-M]:

T2. $f(G; t, z) = f(G/x; t, z) + t^{r(G) - r(G-x)} f(G-x; t, z)$ for any feasible singleton x.

Any greedoid of positive rank must have a feasible singleton (and if $r(G) = 0$, then $f(G) = (z+1)^{|E|}$), so T2 gives us a complete recursion for computing $f(G)$.

If G is a greedoid on E, then $x \in E$ is an *isthmus* if F is feasible precisely when $F \cup \{x\}$ is feasible. We call x a *loop* if x is in no feasible set. The *direct sum* $G_1 + G_2$ of two greedoids $G_1 = (E_1, \mathscr{F}_1)$ and $G_2 = (E_2, \mathscr{F}_2)$ on disjoint sets E_1 and E_2 is the greedoid on the ground set $E_1 \cup E_2$ with feasible sets $\mathscr{F} = \{F_1 \cup F_2 : F_1 \in \mathscr{F}_1 \text{ and } F_2 \in \mathscr{F}_2\}$. The reader can check that if x is an isthmus in G, then $G \cong G/x + I$ and if x is a loop, then $G \cong G - x + L$, where I and L are the one-element greedoids of rank one and zero, respectively. Proposition 3.7 of [G-M] then gives $f(G_1 + G_2) = f(G_1) f(G_2)$; special cases give $f(G) = (t+1) f(G/x)$ if x is an isthmus and $f(G) = (z+1) f(G-x)$ if x is a loop.

An *antimatroid* is a greedoid in which the set of feasible sets is closed under unions. Equivalently, a set system forms the set of feasible sets of an antimatroid if and only if it is accessible and closed under unions. (See Proposition 8.2.7 of [Bj-Z].) A greedoid G on the ground set E is *full* if E is a feasible set. Antimatroids without loops are full greedoids. Antimatroids have appeared in the literature under the names *APS greedoids*, *upper interval greedoids*, *antiexchange greedoids*, *shelling structures*, and *locally free selectors*. As mentioned in the Introduction, a poset P defines a greedoid $G(P)$ in which the feasible sets are the order ideals of P. Since the union of order ideals is again an order ideal, $G(P)$ is an antimatroid.

We conclude this section with a brief discussion of poset operations. Suppose P and Q are posets on disjoint sets. Then we recall the following definitions:

Direct sum. $P + Q$ is a poset on $P \cup Q$ with $x \leqslant y$ in $P + Q$ if either

(a) $x, y \in P$ and $x \leqslant y$ in P or

(b) $x, y \in Q$ and $x \leqslant y$ in Q.

Ordinal sum. $P \oplus Q$ is a poset on $P \cup Q$ with $x \leqslant y$ in $P \oplus Q$ if either

(a) $x, y \in P$ and $x \leqslant y$ in P or

(b) $x, y \in Q$ and $x \leqslant y$ in Q or

(c) $x \in P$ and $y \in Q$.

Direct product. $P \times Q$ is a poset on $\{(x, y) : x \in P \text{ and } y \in Q\}$ with $(x, y) \leqslant (x', y')$ in $P \times Q$ if $x \leqslant x'$ in P and $y \leqslant y'$ in Q.

Ordinal product. $P \otimes Q$ is a poset on $\{(x, y) : x \in P \text{ and } y \in Q\}$ with $(x, y) \leqslant (x', y')$ in $P \otimes Q$ if

(a) $x = x'$ in P and $y \leqslant y'$ in Q or

(b) $x < x'$ in P.

The poset **1** is the one-element poset. A poset is a *series-parallel* poset if it can be built up recursively from **1** by using the operations of direct sum and ordinal sum. The Hasse diagrams of the posets formed by the above operations can all be obtained from the Hasse diagrams of P and Q by straightforward techniques. The reader can consult [S] or work out the details directly. We also point out that greedoid direct sum and poset direct sum are the same operations for poset greedoids. Finally, the *dual* P^* of a poset P is obtained by flipping the Hasse diagram of P, i.e., $x \leqslant y$ in P^* iff $y \leqslant x$ in P.

2. Posets and Antichain Expansions

Let G be an antimatroid on the ground set E with rank function r. We can take advantage of the fact that the union of feasible sets is feasible to obtain an expansion of $f(G; t, z)$ in terms of the feasible sets. We begin with a trivial but useful lemma.

Lemma 2.1. *Let G be an antimatroid and let $A \subseteq E$ with $r(A) = k$. Then there is a unique feasible set B with $B \subseteq A$ and $|B| = k$.*

Proof. If there were two feasible sets B_1 and B_2 of cardinality k contained in A, then their union would be a larger feasible subset of A, contradicting the fact that $r(A) = k$.

We recall one more definition from greedoid theory. Let G be any greedoid and let $A \subseteq E$. Define the *rank closure* of A, denoted $\sigma(A)$, by $\sigma(A) = \{x \in G : r(A \cup \{x\}) = r(A)\}$. The proof of the next theorem is similar to the proof of Proposition 11 of [C-G].

Theorem 2.2. *Let G be an antimatroid. Then* $f(G; t, z) = \sum_{B \text{feasible}} t^{r(G) - |B|}(z+1)^{|\sigma(B)| - |B|}$.

Proof. By 2.1, each subset A contains a unique maximal feasible subset B. In this case, we say A *uses* B. For a given feasible set B, we let $U(B)$

be the collection of all subsets which use B. Note that $r(A)=|B|$ for all $A\in U(B)$. Then we collect terms in T1 as follows:

$$f(G;t,z)=\sum_{B\text{feasible}}\sum_{A\in U(B)} t^{r(G)-|B|}z^{|A|-|B|}=\sum_{B\text{feasible}} t^{r(G)-|B|}z^{-|B|}\sum_{A\in U(B)} z^{|A|}.$$

Now by G2, given a feasible subset B of A, A uses B if and only if for every $x\in A-B$, $B\cup\{x\}$ is not feasible, or equivalently $r(B\cup\{x\})=r(B)$. Thus, $A\in U(B)$ if and only if $A=B\cup C$ for some $C\subseteq\sigma(B)-B$. Hence

$$\sum_{A\in U(B)} z^{|A|}=\sum_{C\subseteq\sigma(B)-B} z^{|B|+|C|}$$
$$=z^{|B|}\sum_{i=0}^{|\sigma(B)|-|B|}\binom{|\sigma(B)|-|B|}{i}z^i=z^{|B|}(z+1)^{|\sigma(B)|-|B|}.$$

Putting these together gives the desired expansion.

When $G(P)$ is a poset greedoid for a poset P, we can use the correspondence between antichains and order ideals to obtain an expansion in purely poset theoretic terms. More precisely, if A is an antichain in a poset P, let $I(A)$ and $I^*(A)$ be the order ideal and order filter, respectively, generated by A. Further, we let $\overline{I(A)}$ and $\overline{I^*(A)}$ be the ideal and filter strictly generated by A, respectively. Thus

$$I(A)=\{x\in P: x\leqslant y \text{ for some } y\in A\},$$
$$I^*(A)=\{x\in P: x\geqslant y \text{ for some } y\in A\},$$
$$\overline{I(A)}=\{x\in P: x< y \text{ for some } y\in A\},$$
$$\overline{I^*(A)}=\{x\in P: x> y \text{ for some } y\in A\}.$$

We remark that I can be regarded as a bijection between the set of all antichains of P (which we denote $A(P)$) and the set of all order ideals (which we denote $F(P)$). I^* gives a bijection between $A(P)$ and the set of all order filters (denoted $F^*(P)$). For ease of notation, we will write $I(x)$, $I^*(x)$, etc. for $I(\{x\})$, $I^*(\{x\})$, etc. when $A=\{x\}$. We will also write $f(P;t,z)$ and $f(P^*;t,z)$ (or simply $f(P)$ and $f(P^*)$) for $f(G(P);t,z)$ and $f(G(P^*);t,z)$, respectively, when $G(P)$ is the poset greedoid corresponding to the poset P. We omit the proof of the next proposition, which follows immediately from 2.2.

PROPOSITION 2.3. *Let P be a poset and P^* be the dual of P. Then*

(a) $f(P;t,z)=\sum_{A\in A(P)} t^{|I^*(A)|}(z+1)^{|\overline{I^*(A)}|}$ *and*
(b) $f(P^*;t,z)=\sum_{A\in A(P)} t^{|I(A)|}(z+1)^{|\overline{I(A)}|}$.

(*Note.* In part (b), $I(A)$ and $\overline{I(A)}$ are computed in P, not P^*.) Thus, we can interpret the polynomial $f(P)$ as a generating function for the number of ordered pairs $(|I^*(A)|, |\overline{I^*(A)}|)$ for all antichains A in the poset.

We can also obtain a deletion–contraction recursion for $f(P)$ when P is a poset. This will require us to interpret deletion and contraction for poset greedoids. Recall that the recursion of T2 is valid only when $\{x\}$ is feasible; i.e., x is a minimal element of the poset. Now assume x is minimal in P. The greedoid contraction $G(P)/x$ is isomorphic to $G(P/x)$, where P/x is the poset on the set $P-\{x\}$ with the inherited partial order. Thus, the Hasse diagram of P/x is obtained from the Hasse diagram of P by simply erasing x and all edges incident with x. Deletion in poset greedoids is more problematic because if $y \in P$ with $x < y$, then y will be in no feasible set of the greedoid deletion $G(P)-x$. In fact, $G(P)-x$ is isomorphic to the greedoid direct sum $G(P-I^*(x))+S$, where the poset $P-I^*(x)$ is obtained from P by restriction to the set $P-I^*(x)$ and S is a collection of $|I^*(x)|-1$ loops. Thus, we can view deletion of x as a process which removes the element x from P and makes each element greater than x a loop in the associated greedoid.

Recall that in a greedoid, an element x is an isthmus if $F \cup \{x\}$ and $F-\{x\}$ are feasible precisely when F is feasible. For a poset, this means x is an isthmus precisely when x is not comparable to any other element in the poset. We also remark that, although our definition of deletion and contraction in posets is consistent with the standard definition of deletion and contraction in greedoids, it is possible to define deletion and contraction in a different way for posets. This is done in [Di], but we will not explore the implications here.

We now apply T2 to $f(G(P))$ when P is a poset. Proposition 2.4 follows from observing that $r(G(P))-r(G(P)-x)=|P|-|P-I^*(x)|=|I^*(x)|$ and that each of the $|\overline{I^*(x)}|$ $(=|I^*(x)|-1)$ loops of $G(P)-x$ contributes a factor of $(z+1)$ to $f(G(P)-x)$.

PROPOSITION 2.4. *Let x be minimal in a poset P. Then*

$$f(P;t,z)=f(P/x;t,z)+t^{|I^*(x)|}(z+1)^{|\overline{I^*(x)}|}f(P-I^*(x);t,z).$$

Note that if $P/x \cong Q/y$ and $P-I^*(x) \cong Q-I^*(y)$ for posets P and Q with $x \in P$ and $y \in Q$, then $|\overline{I^*(x)}|=|\overline{I^*(y)}|$, so 2.4 implies $f(P)=f(Q)$. We will use this simple fact later to produce examples of non-isomorphic posets with the same Tutte polynomial.

It is possible to develop a dual version of the deletion–contraction recursion of 2.4. This time, let x be a *maximal* element of a poset P. We define *upper deletion* and *upper contraction* as follows:

Upper deletion. $P--x$ is the poset on $P-I(x)$, with the inherited partial order.

Upper contraction. $P//x$ is the poset on $P-\{x\}$, with the inherited partial order.

PROPOSITION 2.5. *Let x be a maximal element of a poset P. Then*

$$f(P;t,z)=t(z+1)\,f(P//x)+(1-tz)\,f(P--x).$$

Proof. We use T1 and separate subsets of P into two classes, those subsets which contain x and those which do not:

$$f(P)=\sum_{S:x\in S} t^{|P|-r(S)}z^{|S|-r(S)}+\sum_{S:x\notin S} t^{|P|-r(S)}z^{|S|-r(S)}.$$

Let $r_Q(S)$ denote the rank in the greedoid $G(Q)$ of the subset S. If $x\notin S$, then $r_P(S)=r_{P//x}(S)$, but $|P|=|P//x|+1$. Thus $\sum_{S:x\notin S} t^{|P|-r(S)}z^{|S|-r(S)}=tf(P//x)$.

For the subsets S which contain x, we again separate into two classes: the subsets that contain all of $I(x)$ and those that do not,

$$\sum_{S:x\in S} t^{|P|-r(S)}z^{|S|-r(S)}=\sum_{S:S\supseteq I(x)} t^{|P|-r(S)}z^{|S|-r(S)}$$
$$+\sum_{S:x\in S\not\supseteq I(x)} t^{|P|-r(S)}z^{|S|-r(S)}. \qquad (\#)$$

For the first of the summations on the right-hand side of $(\#)$ (in which $S\supseteq I(x)$), we obtain $r_P(S)=r_{P--x}(S-I(x))+|I(x)|$, $|S|=|S-I(x)|+|I(x)|$ and $|P|=|P--x|+|I(x)|$, so

$$\sum_{S:S\supseteq I(x)} t^{|P|-r(S)}z^{|S|-r(S)}=f(P--x).$$

To analyze the second summation on the right-hand side of $(\#)$ (in which $x\in S\not\supseteq I(x)$), we first write

$$f(P//x)=\sum_{S:S\supseteq \overline{I(x)}} t^{|P//x|-r(S)}z^{|S|-r(S)}+\sum_{S:S\not\supseteq \overline{I(x)}} t^{|P//x|-r(S)}z^{|S|-r(S)}, \qquad (\#\#)$$

where $r(S)=r_{P//x}(S)$ is the rank in $P//x$ and $x\notin S$. For ease of notation, let A and B equal the first and second summations on the right-hand side of $(\#\#)$, respectively. To analyze A (in which $x\notin S\supseteq \overline{I(x)}$), we have $r_P(S\cup\{x\})=r_{P//x}(S)+1$, $|S\cup\{x\}|=|S|+1$, and $|P|=|P//x|+1$, so $A=\sum_{S:S\supseteq I(x)} t^{|P|-r(S)}z^{|S|-r(S)}$, where $r(S)=r_P(S)$ is the rank in P in the latter summation. Thus, by the argument used in analyzing the first summation on the right-hand side of (#),

$$A = f(P - - x).$$

To analyze B (in which $x \notin S \not\supseteq \overline{I(x)}$), we obtain $r_P(S \cup \{x\}) = r_{P//x}(S)$, which gives

$$B = (tz)^{-1} \sum_{S : x \in S \not\supseteq I(x)} t^{|P| - r(S)} z^{|S| - r(S)},$$

where $r(S) = r_P(S)$ is the rank in P in the latter summation. Thus we can rewrite (# #) as

$$f(P//x) = f(P - - x) + (tz)^{-1} \sum_{S : S \not\supseteq I(x)} t^{|P| - r(S)} z^{|S| - r(S)}.$$

Putting the pieces together yields $f(P) = tf(P//x) + f(P - - x) + tz[f(P//x) - f(P - - x)]$, which simplifies to the result.

If we apply 2.5 to P^*, we obtain the following corollary.

COROLLARY 2.6. *Let x be a minimal element of P. Then*

$$f(P^*; t, z) = t(z+1) f((P/x)^*) + (1 - tz) f((P - I^*(x))^*).$$

As an application of 2.4 and 2.5, we consider the following example.

EXAMPLE 2.7. Let Z_n denote the n-element "fence" or "zigzag" poset. Generalizing problem 23(b), page 157 of [S], we wish to find a closed form expression for the generating function $G(x) = \sum_{n \geq 0} f(Z_n^*; t, z) x^n$. (The generating function $F(x) = \sum_{n \geq 0} W_n(q) x^n$ of [S] is an evaluation of $G(x)$, where $W_n(q)$ is the rank generating function of the lattice $J(Z_n)$ of order ideals of Z_n. The *rank-generating function* $R(P; q)$ is defined on a graded poset P of rank n by $R(P; q) = \sum_{i=0}^{n} p_i q^i$, where p_i is the number of elements of rank i in P. Then $f(P^*; q, 0) = R(J(P); q)$.)

Applying 2.4 to $x_2 \in Z_{2n}^*$, we obtain the recursion

$$f(Z_{2n}^*) = f(Z_{2n-1}^*) + t^2(z+1) f(Z_{2n-2}^*), \qquad n \geq 1. \qquad (2.7.1)$$

Applying 2.5 to $x_1 \in Z_{2n+1}^*$, we obtain the recursion

$$f(Z_{2n+1}^*) = t(z+1) f(Z_{2n}^*) + (1 - tz) f(Z_{2n-1}^*), \qquad n \geq 1. \qquad (2.7.2)$$

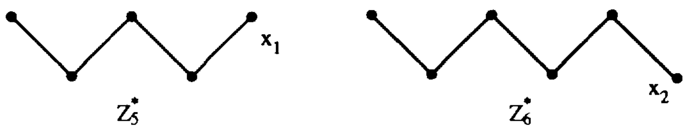


FIGURE 1

(See Fig. 1.) It is now a routine exercise to obtain a closed form expression for $G(x)$. We obtain

$$G(x)=\frac{1+(t+1)x+(t^3(z^2+z)-t^2(z+1))\,x^3}{1-(t^2(z+1)+t+1)\,x^2-(t^3(z^2+z)-t^2(z+1))\,x^4}.$$

This reduces to $F(x)$ when we substitute $t=q$, $z=0$.

We conclude this section with an alternate formulation of $f(P)$ based on activities expansions. In [G-T], an *activities* approach to the Tutte polynomial of a matroid connects several different developments. Other approaches for matroids appear in [Cr, Bj, Da]. A similar approach will work for posets, giving us a unified view of T1, 2.3, and 2.4. We give a short description of this approach now. Let $\bar{\mathbf{P}}$ be the family of posets with loops, i.e., greedoids which are the direct sum of a poset greedoid and greedoid loops. For a minimal element $x\in P\in\bar{\mathbf{P}}$, we use the greedoid definitions of deletion and contraction as in the comments preceding 2.4. Thus the contraction P/x is the same as before, where the loops of P remain loops in P/x, while the deletion $P-x$ is (again) the poset on $P-I^*(x)$ with the addition of $|I^*(x)|-1$ loops (and the loops of P again remain loops in $P-x$). If x is a loop in P, then the deletion $P-x$ is obtained from P by simply removing the loop x. When x is a loop, we *define* P/x to be $P-x$. When x is an isthmus, then deletion and contraction are both defined from before and one can (easily) show that $P-x=P/x$ is again valid.

We can now *define* the Tutte polynomial $f(P;t,z)$ recursively for posets with loops.

DEFINITION 2.8. Let $P\in\bar{\mathbf{P}}$ and let $x\in P$:

(a) $f(P;t,z)=f(P/x;t,z)+t^{|I^*(x)|}f(P-x;t,z)$ if x is minimal in P, x not an isthmus;

(b) $f(P;t,z)=(t+1)\,f(P/x;t,z)$ if x is an isthmus;

(c) $f(P;t,z)=(z+1)\,f(P-x;t,z)$ if x is a loop.

If P has no loops, then it follows immediately from 2.4 that this polynomial is the same Tutte polynomial we have been using throughout this paper.

Now let P be a poset (without loops) and, following [G-T], we extend the partial order to a total order and consider all $2^{|P|}$ different ways of resolving P by deleting and contracting the elements of P in the order given by the linear extension. (*Note.* This ensures that all elements deleted and contracted will be minimal or loops at the time they are deleted or contracted.) To each such resolution, we associate the set S of elements which are contracted during the resolution. For a given subset S, we let $P_k \in \bar{\mathbf{P}}$ be the poset (possibly with loops) obtained by either deleting (if the element is not in S) or contracting (if it is in S) the first k elements of P in the order given by the linear extension. We say $x \in P$ is an *eventual isthmus* (resp., *eventual loop*) with respect to S if x is an isthmus (resp., loop) in P_{k-1}, if x is the k^{th} element of the linear extension of P. If x is neither an eventual isthmus nor loop, we say x is *ordinary*. For an eventual isthmus $x \in S$, we call x an *internal eventual isthmus*, or simply an *internal isthmus* of S; if $x \notin S$, we call x an *external eventual isthmus*, or, more simply, an *external isthmus* of S. Similarly, we define *internal* and *external loops* and *internal* and *external ordinary* elements with respect to S. Let $i(S)$ denote the set of eventual isthmuses of S, $ei(S)$ denote the set of external isthmuses of S, and so on.

The main result concerning activities is the following theorem. All other expansions of the Tutte polynomial discussed here can be obtained from this result (see 2.11).

THEOREM 2.9. *Let a and b be any two elements of the* polynomial *ring* $\mathbf{Z}[t, z]$, *and let P be a poset (without loops). Then* $f(P; t, z) = \sum_{S \subseteq P} a^{|ii(S)|}(t+1-a)^{|ei(S)|}\, b^{|il(S)|}(z+1-b)^{|el(S)|}\, t^{|eo(S)|+|l(S)|}$.

Proof. Since $P - x = P/x$ if x is an isthmus or a loop, we can replace 2.8(b) and 2.8(c) by

(b′) $f(P) = af(P/x) + (t+1-a)\,f(P-x)$ if x is an isthmus;

(c′) $f(P) = bf(P/x) + (z+1-b)\,f(P-x)$ if x is a loop.

Consider the process of calculating $f(P)$ recursively using 2.8 (with (b′) and (c′) replacing (b) and (c)), applying the recursion in the order specified by the linear extension. The final result of the calculation is to express $f(P)$ as the sum of $2^{|P|}$ terms; the term obtained by contracting the elements of a subset S and deleting the elements of $P - S$ will be

$$a^{|ii(S)|}(t+1-a)^{|ei(S)|}\, b^{|il(S)|}(z+1-b)^{|el(S)|}\, t^{m},$$

where m will now be determined. We get a contribution to m precisely when we perform deletion in 2.8(a); this only occurs when $x \in eo(S)$. Thus $m = \sum_{x \in eo(S)} |I^*(x)|$. Now this sum counts all the loops created as well as all external ordinary elements. Since each loop is created exactly once, $m = |eo(S)| + |l(S)|$ and the theorem follows.

The next proposition will help us to use 2.9 to generate different expansions. We omit the straightforward proof.

PROPOSITION 2.10. *Let P be a poset and $S \subseteq P$. Then $l(S) = \overline{I^*(P-S)}$.*

EXAMPLE 2.11. We will use various choices of a and b in 2.9 to derive expansions of $f(P)$:

(a) Set $a = 1$ and $b = z$. Then we obtain $f(P) = \sum_{S \subseteq P} t^{|ei(S)| + |eo(S)| + |l(S)|} z^{|il(S)|}$. Now $|i(S)| + |l(S)| + |o(S)| = |P|$, so $|ei(S)| + |eo(S)| + |l(S)| = |P| - (|ii(S)| + |io(S)|) = |P| - r(S)$, and $|il(S)| = |S| - r(S)$. Thus this expansion is identical to T1, our original definition of the Tutte polynomial.

(b) Set $a = 1$ and $b = 0$. Then the subset S which will contribute to the sum if and only if $il(S) = \varnothing$, which is equivalent to S being an order ideal (by 2.10). If S is an order ideal, then $|el(S)| = |\sigma(S)| - |S|$, where $\sigma(S) = \{x \in P : r(S \cup \{x\}) = r(S)\}$ is the rank closure (see comments preceding 2.2). This expansion is then the feasible set expansion given in Theorem 2.2.

(c) Set $a = 1$ and $b = z + 1$. We can use 2.10 and obtain the antichain expansion of 2.3(a). We leave details to the reader.

(d) Set $a = t + 1$ and $b = 0$. (This evaluation corresponds to the basis expansion of the Tutte polynomial of a matroid.) Then the corresponding expansion has the form $f(P) = \sum_{S \subseteq P} (t+1)^{|ii(S)|} (z+1)^{|\sigma(S)| - |S|} t^{|P| - |S|}$, where the sum is taken over all order ideals S with $ei(S) = \varnothing$. This expansion has the advantage of having a relatively small index set.

Many other evaluations are possible, although most of these seem difficult to interpret in terms of familiar subsets of a poset. We leave the development of additional expansions to the interested reader.

3. EXAMPLES, APPLICATIONS, AND EVALUATIONS

We begin with a minimal example of non-isomorphic posets which have the same polynomial.

EXAMPLE 3.1. Let P and Q be the two (non-isomorphic) posets with Hasse diagrams shown in Fig. 2. Then $P/x \cong Q/y$ and $P - I^*(x) \cong Q - I^*(y)$, so, by 2.4, we have $f(P) = f(Q)$. The reader can check that

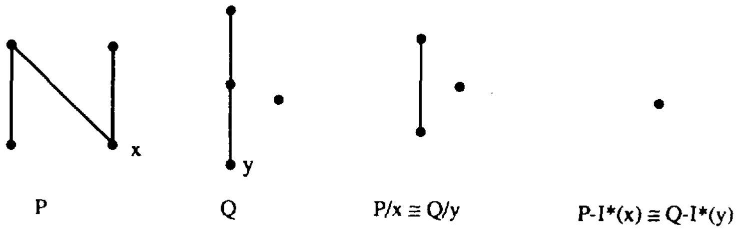


FIGURE 2

$f(P) = f(Q) = t^4(z+1)^2 + t^3[(z+1)^2 + (z+1)] + t^2[(z+1)+1] + 2t + 1 = (t+1)[t^3(z+1)^2 + t^2(z+1) + t + 1]$. The reader can also check that this example is minimal since any poset P on three or fewer elements is uniquely determined by $f(P)$.

A direct consequence of 3.1 is the fact that any poset invariants which differ on P and Q are invariants which cannot be determined from the polynomial. We list some of these invariants here:

3.1.1. Since the longest chain in P has length two and the longest chain in Q has length three, $f(P)$ does not determine the length of the longest chain of P.

3.1.2. P has a total of eight chains (including the empty chain) and Q has nine, so $f(P)$ does not determine the total number of chains in P.

3.1.3. P has 10 multichains of length three and Q has 11, so $f(P)$ does not in general determine the number of multichains of length k.

3.1.4. P has five linear extensions and Q has four, so $f(P)$ does not determine the total number of linear extensions of P. In the language of [S], $f(P)$ does not determine the size of the Jordan–Hölder set. Consequently, the order polynomials of P and Q are different, so $f(P)$ cannot determine the order polynomial of P.

3.1.5. Since the Hasse diagram of P is connected (as a graph) and the Hasse diagram of Q is not, $f(P)$ does not determine whether or not the Hasse diagram is connected. We point out, however, that if $f(P)$ is irreducible over $\mathbf{Z}[t, z]$, then the Hasse diagram of P must be connected. This follows from the direct sum properties of the polynomial. Also, since Q has an isthmus and P does not, $f(P)$ cannot determine whether P has an isthmus.

3.1.6. Since Q is a series-parallel poset and P is not, $f(P)$ does not determine whether or not P is series-parallel.

3.1.7. Since P has three maximal chains and Q has two, the polynomial cannot determine the number of maximal chains.

We now turn our attention to some poset invariants which *can* be determined from $f(P)$.

PROPOSITION 3.2. *Let P and Q be two posets with $f(P)=f(Q)$. Then*

(a) *P and Q have the same number of order ideals of size k for all k;*

(b) *P and Q have the same number of order filters of size k for all k;*

(c) *P and Q have the same number of antichains of size k for all k, and hence the same width;*

(d) $f(P; t, -1)=f(Q; t, -1)=(t+1)^M$, *where M is the number of maximal elements of P, so P and Q have the same number of maximal elements;*

(e) *P and Q have the same number of minimal elements.*

Proof. (a), (b) Since complementation defines a bijection between order ideals and order filters, (a) and (b) are clearly equivalent. Now (a) follows from noting that, by Theorem 2.2, the coefficient of $t^{|P|-k}$ in $f(P; t, 0)$ is just the number of order ideals of size k in P.

(c) Note that, by Proposition 2.3(a), the coefficient of t^k in $f(P; t, t^{-1}-1)$ is the number of antichains of size k in P, and the width of a poset is just the size of the largest antichain.

(d) Clearly $A \subseteq P$ is an antichain such that $\overline{I^*(A)} = \emptyset$ if and only if A is a subset of the set of maximal elements of P. The result follows immediately from 2.3(a).

(e) This is a special case of (a).

Many other coefficients and evaluations are of interest, but we will not explore these in depth. For example, the coefficient of $t^{|P|}$ in $f(P; t, z)$ is $(z+1)^{|P|-m}$, where m is the number of minimal elements of P. Further, various restrictions of f to one-variable polynomials are of interest for posets and, more generally, for arbitrary greedoids.

The evaluation $f(P; t, 0)$ is an interesting invariant on its own, since it can be thought of as a generating function of the sizes of all order ideals of P (3.2a). The next example shows that this invariant is strictly weaker than $f(P; t, z)$.

EXAMPLE 3.3. Let P and Q be the posets of Fig. 3. Then $f(P; t, 0) = (t+1)(t^4+t^3+2t^2+t+1)$ and $f(Q; t, 0) = (t^2+t+1)(t^3+t^2+t+1)$. Factoring these products yields $f(P; t, 0) = f(Q; t, 0) = (t+1)(t^2+1)(t^2+t+1)$. But since P has an antichain of size 3 and Q does not, $f(P; t, z) \neq f(Q; t, z)$ by 3.2(c).

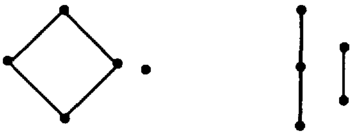


FIGURE 3

We also point out that similar examples exist for arbitrary antimatroids, matroids, and rooted trees, as well as for posets which are not self-dual.

The next proposition is proven in [G-M].

PROPOSITION 3.4. *Let G be a greedoid with* $|G| = n$ *and* $r(G) = r$:

(a) $f(G; t, t^{-1}) = t^{r-n}(t+1)^n$;

(b) $f(G; 0, y-1) = \lambda_G(y)$, *where* $\lambda_G(y)$ *is the greedoid polynomial of* [Bj-Z].

The remainder of this section is devoted to three applications. We begin with a polynomial which Stanley considers in Chapter 3 of [S].

EXAMPLE 3.5. Let P be a poset, and, as before, let $A(P)$ denote the set of all antichains of P. Then set $G_P(x, y) = \sum_{A \in A(P)} x^{|I(A)|} y^{|A|}$. Then $G_P(x, y)$ is the poset polynomial defined on page 158 in problem 26 of [S]. (Stanley attributes this problem to M. Haiman—see page 182 of [S].) We now show that $G_P(x, y)$ is essentially equivalent to $f(P^*)$.

PROPOSITION 3.6. $G_P(x, y) = f(P^*; xy, y^{-1} - 1)$.

Proof. By 2.3(b), we have $f(P^*; xy, y^{-1} - 1) = \sum_{A \in A(P)} (xy)^{|I(A)|} (y^{-1})^{|I(A)| - |A|}$ since $|A| + |\overline{I(A)}| = |I(A)|$. This immediately reduces to $G_P(x, y)$.

Part (a) of problem 26 [S] concerns the *ordinal product* of two posets, which we explore in Section 4. Part (b) of the problem shows $G_P(q, q^{-1}(q-1)) = q^{|P|}$, which now follows from 3.6 and 3.4(a).

Our second application concerns the concept of the dimension of a poset. Our treatment follows [K-T], which gives a nice introduction to this very well-studied and interesting subject. The *dimension* of P, $\dim(P)$, can be defined as the minimum number of linear extensions of P whose intersection gives P. In view of 3.1.4 and the definition of dimension, it is not surprising that $f(P)$ cannot determine $\dim(P)$, which is the point of the next example.

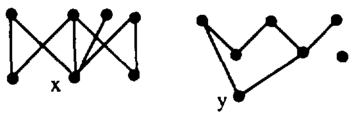


FIGURE 4

EXAMPLE 3.7. Let P and Q be as shown in Fig. 4. The reader can check that $\dim(P) = 3$ and $\dim(Q) = 2$. (The poset P is a *three-irreducible* poset, i.e., removing any element from P reduces the dimension to two. In the notation of [K-T], which is due to Kelly, P is denoted CX_2.) As in 3.1, we again have $P/x \cong Q/y$ and $P - I^*(x) \cong Q - I^*(y)$, so $f(P) = f(Q)$. Hence $f(P)$ cannot be used to determine $\dim(P)$.

Despite 3.7, we can say something about $\dim(P)$ from $f(P)$. For example, Theorem 5.5 of [K-T] asserts that $\dim(P) \leqslant \text{width}(P)$, where $\text{width}(P)$ is the size of the largest antichain in P. (This result is due to Hiraguchi.) Since the size of the largest antichain equals $\max\{a - b : t^a(z+1)^b$ has a non-zero coefficient in the expansion 2.3(a) for $f(P)\}$, $f(P)$ also gives this upper bound on dimension. Other results connecting antichains in P and bounds on $\dim(P)$ appear in [K-T]. Evidently, any such result can be rephrased in terms of $f(P)$. We leave further connections of this kind to the interested reader.

We conclude this section with a result (Theorem 3.10) which extends Theorem 2.8 of [G-M]. We define a class of posets $\mathscr{C}$ with the property that if P and Q are in $\mathscr{C}$, then $f(P) = f(Q)$ if and only if $P \cong Q$.

DEFINITION 3.8. Define the class $\mathscr{C}$ recursively as the smallest class of posets which satisfies:

(a) $\mathbf{1} \in \mathscr{C}$.

(b) If $P, Q \in \mathscr{C}$, then $P + Q \in \mathscr{C}$ (i.e., $\mathscr{C}$ is closed under direct sum).

(c) If $P \in \mathscr{C}$, then $\mathbf{1} \oplus P \in \mathscr{C}$ (i.e., $\mathscr{C}$ is closed under "cupping").

(d) If $P \in \mathscr{C}$, then $P \oplus \mathbf{1} \in \mathscr{C}$ (i.e., $\mathscr{C}$ is closed under "capping").

We note that $\mathscr{C}$ is a subclass of the class of series-parallel posets and that 14 of the 16 non-isomorphic posets on four elements are in $\mathscr{C}$.

To prove Theorem 3.10, we will need the following results on irreducibility of $f(P; t, z)$ over the domain $\mathbf{Z}[t, z]$. Part (a) of 3.9 below appears as Proposition 2.7 in [G-M] and part (b) appears as (half of) Proposition 8 of [C-G]. Although the statements of these results in [G-M] and [C-G] are given for rooted trees, both proofs carry over to the generality of full greedoids.

PROPOSITION 3.9. *Let G be a full greedoid.*

(a) *If G has a unique feasible singleton, then* $f(G; t, z)$ *is irreducible over* $\mathbf{Z}[t, z]$.

(b) *If G has a unique feasible set of size* $|G| - 1$, *then* $f(G; t, z)$ *is irreducible over* $\mathbf{Z}[t, z]$.

THEOREM 3.10. *Suppose* $P_1, P_2 \in \mathscr{C}$. *Then* $f(P_1) = f(P_2)$ *if and only if* $P_1 \cong P_2$.

Proof. We show that if $P \in \mathscr{C}$, then P is uniquely determined by $f(P)$, which is clearly equivalent to the result. We proceed by induction on $|P|$. The result is trivial if $|P| = 1$, so assume $|P| \geqslant 2$. From the definition of $\mathscr{C}$, one of the following must hold (note that we can determine which case holds solely from $f(P)$; also Cases 1 and 2 can hold simultaneously):

Case 1. $P = \mathbf{1} \oplus Q$ for some poset $Q \in \mathscr{C}$. Then $Q = P/x$, where x is the unique minimal element of P. By 2.4, $f(Q) = f(P) - t^{|P|}(z+1)^{|P|-1}$, and so by induction, we can uniquely determine Q. Hence, we can reconstruct the poset $\mathbf{1} \oplus Q = P$.

Case 2. $P = Q \oplus \mathbf{1}$ for some poset $Q \in \mathscr{C}$. This time, $Q = P//x$, where x is the unique maximal element of P. By 2.5, $f(Q) = (t(z+1))^{-1} [f(P) + tz - 1]$, so again by induction we can uniquely determine the poset Q. Since $P = Q \oplus \mathbf{1}$, we can reconstruct P.

Case 3. $P = R_1 + R_2 + \cdots + R_n$ for some $n \geqslant 2$, where each $R_i \in \mathscr{C}$ is irreducible (i.e., no R_i can be expressed as a non-trivial direct sum). By 4.1(a), we obtain $f(P) = f(R_1) f(R_2) \cdots f(R_n)$. By definition of $\mathscr{C}$, we can write either $R_i = \mathbf{1} \oplus Q_i$ or $R_i = Q_i \oplus \mathbf{1}$ for some $Q_i \in \mathscr{C}$, $1 \leqslant i \leqslant n$. In either case, by 3.9, $f(R_i)$ is irreducible over $\mathbf{Z}[t, z]$ for all i. Hence, we reconstruct P as follows:

1. Factor $f(P)$ completely over $\mathbf{Z}[t, z]$.

2. Reconstruct each component R_i inductively from the irreducible factor $f(R_i)$.

3. Form P as the direct sum of the R_i.

This completes the proof.

4. Poset Operations, Duality, and More Counterexamples

We begin by considering the behavior of $f(P)$ under the standard poset operations.

PROPOSITION 4.1. *Let P and Q be two posets.*

(a) *Direct sum,* $f(P+Q)=f(P)f(Q)$;

(b) *Ordinal sum,* $f(P\oplus Q)=f(Q)+(t(z+1))^{|Q|}(f(P)-1)$;

(c) *Ordinal product,* $f(P\otimes Q;t,z)=f(P;f(Q)-1,(t(z+1))^{|Q|}(f(Q)-1)^{-1}-1)$.

Proof. (a) $P+Q$, considered as a poset greedoid, is the greedoid direct sum of the poset greedoids P and Q. Now this is a special case of Proposition 3.7 of [G-M].

(b) Let A be an antichain in $P\oplus Q$. Since every element of P is less than every element of Q in $P\oplus Q$, A is either a non-empty antichain entirely contained in P or entirely contained in Q.

Case 1. $A\neq\varnothing$ is an antichain in P. Then $|I^*(A)|$ (in $P\oplus Q$) $=|Q|+|I^*(A)|$ (in P) and $|\overline{I^*(A)}|$ (in $P\oplus Q$) $=|Q|+|\overline{I^*(A)}|$ (in P).

Case 2. A is an antichain in Q. Then the computations of $|I^*(A)|$ and $|\overline{I^*(A)}|$ in $P\oplus Q$ are exactly the same as in Q.

Using 2.3(a) and putting the two cases together gives

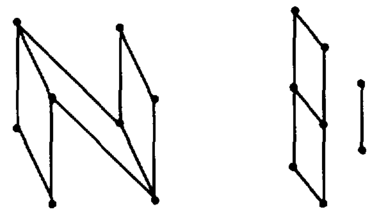
$$\begin{aligned} f(P\oplus Q;t,z) &= \sum_{A\in A(P\oplus Q)} t^{|I^*(A)|}(z+1)^{|\overline{I^*(A)}|} \\ &= (t(z+1))^{|Q|}\sum_{\varnothing\neq A\in A(P)} t^{|I^*(A)|}(z+1)^{|\overline{I^*(A)}|} \\ &\quad + \sum_{A\in A(Q)} t^{|I^*(A)|}(z+1)^{|\overline{I^*(A)}|}. \end{aligned}$$

The formula now follows from 2.3(a).

(c) This formula follows from 3.6 and part (a) of problem 26, page 158 of [S]. The reader can also provide a direct proof by considering antichains in $P\otimes Q$ and arguing along lines similar to the proof of (b).

The omission of a formula for direct products in 4.1 is a consequence of the next example.

EXAMPLE 4.2. Let P and Q be as in Example 3.1 and let R be a two-element chain. Then $P\times R$ has an antichain of size four, but $Q\times R$ does not (see Fig. 5). Hence, by Proposition 3.2(c), $f(P\times R)\neq f(Q\times R)$ even though



P×R Q×R

FIGURE 5

$f(P)=f(Q)$. Since P and Q were minimal in 3.1 and since R has two elements, this example is also minimal. Consequently, $f(P \times Q)$ cannot be expressed as $F(f(P), f(Q))$ for any function $F: \mathbf{Z}[t, z] \times \mathbf{Z}[t, z] \to \mathbf{Z}[t, z]$.

We now turn our attention to duality. In Theorem 4.5, we show $f(P^*)$ is completely determined by $f(P)$, so if $f(P)=f(Q)$, then $f(P^*)=f(Q^*)$. The next lemma shows how we can determine $f(P)$ and $f(P^*)$ from the distributive lattice of order ideals $J(P)$ of a poset P. That $J(P)$ is distributive (which follows from the fact that the order ideals of P are closed under union and intersection) is crucial for the proof of 4.5. The fact that $P \to J(P)$ defines a bijection between posets and distributive lattices (the *fundamental theorem for finite distributive lattices*) is due to Birkhoff [Bi, Theorem III.3]. This theorem and related material can be found on page 106 of [S], for example.

LEMMA 4.4. *Let $J(P)$ be the distributive lattice of order ideals of a poset P. Let $u(i,j)$ be the number of elements of $J(P)$ of rank i which cover exactly j elements and let $v(i,j)$ be the number of elements of $J(P)$ of rank i which are covered by exactly j elements. Then*

(a) $f(P)=\sum_{ij} v(i,j)\, t^{|P|-i}(z+1)^{|P|-i-j}$ *and*

(b) $f(P^*)=\sum_{ij} u(i,j)\, t^i(z+1)^{i-j}$.

Proof. (a) Recall the elements of $J(P)$ correspond to the order ideals of P. The elements of $J(P)$ which *cover* a given element S of rank i are the subsets of the form $S \cup \{x\}$, where $x \in A$ and A is the antichain consisting of the minimal elements of $P-S$. Then $I^*(A)=P-S$ and $\overline{I^*(A)}=P-(S \cup A)$. For $|S|=i$ and $|A|=j$, we find that $v(i,j)$ is the coefficient of $t^{|P|-i}(z+1)^{|P|-i-j}$ in the expansion 2.3(a).

(b) The elements of $J(P)$ which *are covered* by a given element S of rank i are the subsets of the form $S-\{x\}$, where $x \in A$ and A is the antichain consisting of the maximal elements of S. Now $I(A)=S$ and $\overline{I(A)}=S-A$. As in (a), for $|S|=i$ and $|A|=j$, we find that $u(i,j)$ is the coefficient of $t^i(z+1)^{(i-j)}$ in the expansion 2.3(b).

We can now prove the main result of the section.

THEOREM 4.5. *Let P be a poset. Then $f(P^*)$ is completely determined by $f(P)$, i.e., if $f(P)=f(Q)$, then $f(P^*)=f(Q^*)$.*

Proof. By 4.4, knowing $f(P)$ is equivalent to knowing $v(i,j)$ and knowing $f(P^*)$ corresponds to knowing $u(i,j)$ for all $0 \leqslant i, j \leqslant |P|$. We will now show that we can determine each $u(i,j)$ from the collection of all the $v(i,j)$.

From problem 21, page 157 of [S], we obtain for all $i \geqslant j \geqslant 0$,

$$\sum_{k \geqslant 0} u(i,k)\binom{k}{j} = \sum_{k \geqslant 0} v(i-j,k)\binom{k}{j}. \tag{*}$$

(Each side of this equation counts Boolean algebras in $J(P)$ with maximum element of rank i and minimum element of rank $i-j$.) We now construct the $u(i,j)$ by induction. Clearly, $u(0,0)=1$ and $u(1,1)=$ the number of minimal elements of P ($=$ the number of order ideals of size one). Now let $i>1$ and assume we know $u(k,j)$ for all $k<i$ and all j. Let $m \leqslant i$ be the maximum positive integer such that $v(i-m,n)>0$ for some $n \geqslant m$. Then, from (*), $u(i,m)=\sum_{k \geqslant m} v(i-m,k)\binom{k}{m}$ and $u(i,j)=0$ for all $j>m$. To determine $u(i,j)$ for $j<m$, assume inductively that we are given $j<m$ and that we have determined $u(i,k)$ for all k such that $j<k \leqslant m$. Then we can solve for $u(i,j)$ in Eq. (*). This completes the proof.

We point out that the result is trivial for $f(P;t,0)$, since $f(P^*;t,0)=t^{|P|}f(P;t^{-1},0)$. The proof of 4.5 makes use of the correspondence between P and $J(P)$, which we now examine from the viewpoint of the polynomial. The next example shows that we cannot determine $f(J(P))$ from $f(P)$.

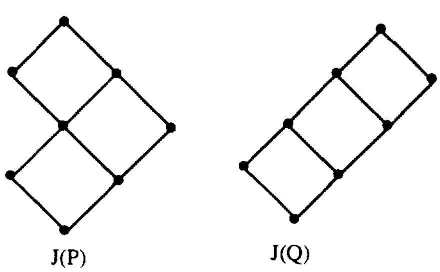


FIGURE 6

EXAMPLE 4.6. Let $J(P)$ and $J(Q)$ be the lattices arising from posets P and Q of Example 3.1 (see Fig. 6). Then

$$f(J(P); t, 0) = t^8 + t^7 + 2t^6 + 2t^5 + 2t^4 + 2t^3 + 2t^2 + t + 1,$$
$$f(J(Q); t, 0) = t^8 + t^7 + 2t^6 + 2t^5 + 3t^4 + 2t^3 + 2t^2 + t + 1.$$

Hence, $f(P) = f(Q)$, but $f(J(P)) \neq f(J(Q))$.

5. FEASIBLE ISOMORPHISM

DEFINITION 5.1. Let G and H be greedoids of rank r on the same ground set $E = \{1, 2, ..., n\}$. Let $F_k(G)$ and $F_k(H)$ denote the collection of all feasible sets of size k in G and H, respectively. A *feasible isomorphism* from G to H is family of permutations π_k of E, $1 \leqslant k \leqslant r$, such that the induced set map $\bar{\pi}_k$ defines a bijection from $F_k(G)$ to $F_k(H)$. If there is a feasible isomorphism from G to H, we say G and H are *feasibly isomorphic*.

Clearly, if G and H are isomorphic, then they are feasibly isomorphic. For matroids, the converse is also true since matroids can be characterized by their bases. In general, this is false for greedoids. As an example, consider the posets P and Q from Example 3.1. Then the reader can check that P and Q are feasibly isomorphic by listing all the order ideals of P and Q.

Although it is also obvious that if P and Q are feasibly isomorphic, then $f(P; t, 0) = f(Q; t, 0)$, feasible isomorphism is substanstially stronger than this. The next theorem is the main result of this section.

THEOREM 5.2. *If G and H are feasibly isomorphic, then $f(G; t, z) = f(H; t, z)$.*

Proof. Let a_{ij} and b_{ij} be the number of subsets of E of rank i and cardinality j in G and H, respectively. Then a_{ij} and b_{ij} are the coefficients of $t^{r-i}z^{j-i}$ in the expansions T1 for $f(G)$ and $f(H)$, respectively. We will show $a_{ij} = b_{ij}$ for all i and j, thus proving $f(G; t, z) = f(H; t, z)$.

Clearly, $a_{jj} = b_{jj}$ for all j from the definition of feasible isomorphism. Now fix i and j with $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant n$, and $i < j$. Now use the permutation π_i on E to replace H by H', so that G and H' have exactly the same feasible sets of size i and $H \cong H'$. Then the number of subsets of cardinality j having rank *at least* i is the same in G as H'. (This is just the number of subsets of size j in E which contain some feasible set of size i, which are identical in G and H'.) Thus $\sum_{k=i}^{j} a_{kj} = \sum_{k=i}^{j} b_{kj}$. By the same argument, $\sum_{k=i+1}^{j} a_{kj} = \sum_{k=i+1}^{j} b_{kj}$. Subtracting these two equations gives $a_{ij} = b_{ij}$ and we are done.

The proof of the next proposition is obvious from the definition of feasible isomorphism.

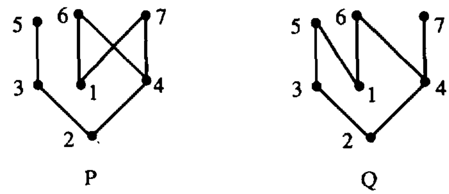


FIGURE 7

PROPOSITION 5.3. *If P and Q are feasibly isomorphic posets, then P^* and Q^* are feasibly isomorphic.*

Proposition 5.3 holds for any class of greedoids closed under duality, e.g., full Gaussian elimination greedoids. We also note that the converse of 5.2 with 5.3 would give another proof of 4.5; however, the next example shows the converse to 5.2 is false for posets.

EXAMPLE 5.4. Let P and Q be the posets of Fig. 7. Then $P/1 \cong Q/1$ and $P - I^*(1) \cong Q - I^*(1)$, so $f(P) = f(Q)$. The order ideals of size 4 in P are $\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 6\}, \{1, 2, 4, 7\}, \{2, 3, 4, 5\}\}$ and the order ideals of size 4 in Q are $\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 6\}, \{1, 2, 4, 7\}, \{2, 3, 4, 7\}\}$. Then no permutation π_4 will map one family onto the other, so P and Q are not feasibly isomorphic.

Although feasible isomorphism does not imply isomorphism in general, there are some classes of greedoids for which this implication is valid. In particular, the following result follows immediately from 5.2.

PROPOSITION 5.5. *Let $\mathscr{C}$ be a class of greedoids with the property that if G_1 and G_2 are non-isomorphic greedoids in $\mathscr{C}$, then $f(G_1) \neq f(G_2)$. Then G_1 and G_2 are isomorphic if and only if they are feasibly isomorphic.*

In particular, the class $\mathscr{C}$ of posets defined in 3.8 satisfies the hypothesis of 5.5. The interested reader can try to construct a direct proof that feasible isomorphism is equivalent to isomorphism in this case.

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