MATROID REPRESENTATION, GEOMETRY AND MATRICES

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1. Introduction

The connections between algebra and finite geometry are very old, with theorems about configurations of points dating to ancient Greece. In these notes, we will put a matroid theoretic spin on these results, with matroid representations playing the central role.

Recall the definition of a matroid via independent sets $\mathcal{I}$.

Definition 1.1. Let E be a finite set and let $\mathcal{I}$ be a family of subsets of E. Then the family $\mathcal{I}$ forms the *independent sets of a matroid* M if:

(I1) $\mathcal{I} \neq \emptyset$
(I2) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
(I3) If $I, J \in \mathcal{I}$ with $|I| < |J|$, then there is some element $x \in J - I$ with $I \cup \{x\} \in \mathcal{I}$

We begin with an example to show how a finite collection of vectors can be used to create a *picture* of the matroid.

Example 1.2. Let N be the following 3×5 matrix:

$$N = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Our immediate goal is to produce a configuration that represents the linear dependences of the columns of N. We use the following procedure:

- Each column vector will be represented by a point;
- If three vectors u, v and w are linearly dependent, then the corresponding three points will be collinear.

Notice that columns a, b and c are linearly dependent. That means the corresponding three points will be collinear. This process shows that the matroid M corresponding to the matrix N is what we have depicted in Fig. 1.

You can view the picture as a dimension reducing procedure (see Fig. 2): The rank of the matroid is 3; this is simply the size of the largest independent set. This is (conveniently) also the rank of the matrix. But the picture in Figure 1 is 2-dimensional, even though the vectors live in $\mathbb{R}^3$. This works generally for rank 3 matrices: Draw the vectors and find a plane in free position that meets each of the lines determined by the vectors. The picture of the matroid (which we'll simply call "the matroid") will then correspond to the points of intersection of the lines with the distinguished plane.

There are three key questions to consider:

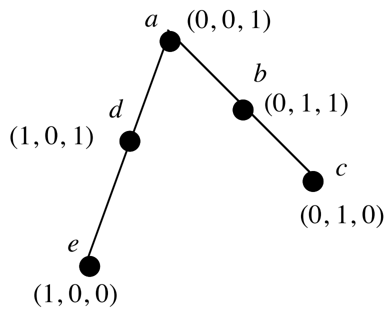

FIGURE 1. The matroid corresponding to the matrix N.

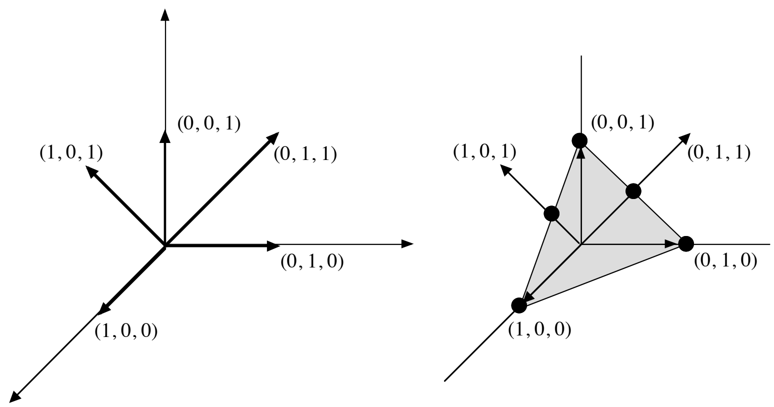

FIGURE 2. Projecting the column vectors of N onto the plane $x + y + z = 1$.

Q1. Does any configuration of points and lines give a matroid?
Q2. Does every subset of vectors give rise to a matroid, i.e., do the subsets of linearly independent column vectors of a matrix satisfy (I1), (I2) and (I3)?
Q3. Does every matroid arise as the linear dependences of a collection of vectors?

The first question is easy to deal with, depending on what we mean by "configuration."

Proposition 1.3. *A finite set of points and lines in the plane is a matroid if and only if any pair of lines meet in at most one point.*

Exercise 1. Prove the proposition. (Hint: For one direction: If two lines meet in two (or more) points, there must be four points a, b, c and d satisfying: a, b, c are collinear, a, b, d are collinear, but a, b, c, d are not all on one line. Then apply (I3) to appropriate independent sets I and J to get a contradiction.)

The answer to the second question is yes - indeed, the name "matroid" is derived from the word " matrix," and a matroid can be thought of as an abstraction of linear dependence.

Theorem 1.4. *Let E be a finite set of vectors in a vector space V, and let $\mathcal{I}$ be those subsets of E that are linearly independent. Then $\mathcal{I}$ is the family of independent sets of a matroid.*

We don't include a proof of Theorem 1.4 here, but the interested reader should try to use the fundamental properties of matrix rank to prove the theorem as an exercise.

The answer to Q3 is no - we can construct matroids that do not arise from column dependences for any matrix (see Example 2.2). This is extremely important, and it lies at the heart of the subject. If every matroid came from a matrix, then all of matroid theory would be subsumed under linear algebra.

We conclude the introduction with another example, reversing the procedure from Example 1.2.

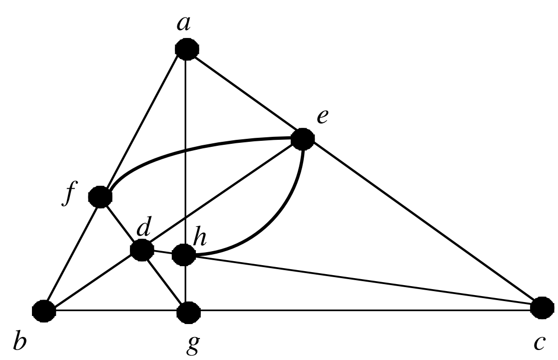

FIGURE 3. The Matroid $M_{\sqrt{-3}}$.

Example 1.5. (Brylawski and Kelly [2]) Let M be the matroid pictured in Fig. 3, where the points efh are collinear. You can check this is a matroid by Prop. 1.3. We will see the geometry is intricately connected to an algebraic condition. Our goal in this example is to create a matrix whose column (linear) dependences match the matroid dependences. We assume the following (all of which can be verified):

- The first non-zero entry in every column of N can be taken to be 1.
- Since $r(M) = 3$, we may take N to be a 3×8 matrix.
- Since abc is a basis, we may take the first three columns of N to be an identity matrix.
- Since d is not on any of the lines determined by a, b and c, we can also take d to be the column vector $[1, 1, 1]^T$.

Here's what we have so far:

$$A = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & \cdots \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \end{array} \right] \end{array}.$$

We need to determine the coordinates for the remaining points in M. This will have the flavor of a puzzle.

- Since e is on the line determined by a and c, we know e is a linear combination of $[1, 0, 0]^T$ and $[0, 0, 1]^T$, so the second coordinate is 0.
- Since e is also on the line through b and d, the first and last coordinates of e must be equal.

Putting these together gives $e = [1,0,1]^T$. Continuing in this way, we determine the coordinates for the remaining points. At each stage, we use the coordinates of the points already assigned to determine restrictions on the coordinates of point in question.

- f is on the line through a and b, so $f = [1, x, 0]^T$, where x is temporarily undetermined.
- g is on lines bc and df. The first of these lines forces $g = [0, 1, y]^T$, where y is undetermined. The line dfg forces the determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & x & 1 \\ 1 & 0 & y \end{vmatrix} = 0,$$

  so $xy + 1 - y = 0$. This forces $y = \frac{1}{1-x}$.
- h is the last point, and it's on three lines we can use to determine its coordinates: agh, cdh and efh. The line cdh forces the first two coordinates of h to be equal, so we can assume $h = [1, 1, z]$.

$$A = \begin{array}{c} \begin{matrix} \scriptstyle a & \scriptstyle b & \scriptstyle c & \scriptstyle d & \scriptstyle e & \scriptstyle f & \scriptstyle g & \scriptstyle h \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & \frac{1}{1-x} & z \end{bmatrix} \end{array}.$$

  We have 2 more dependences to consider: agh and efh. One of these will determine the value of z, and the other will force x to satisfy some relation. First, for agh, we get the determinant

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & \frac{1}{1-x} & z \end{vmatrix} = 0,$$

  so $z = \frac{1}{1-x}$. Finally, for efh, we have the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 1 \\ 1 & 0 & \frac{1}{1-x} \end{vmatrix} = 0,$$

  which gives $x^2 - x + 1 = 0$.

To sum up, at each stage of this process, we made the most general choice for coordinates possible (under mild assumptions). But this forced us to choose a value for x that satisfies the quadratic equation $x^2 - x + 1 = 0$. Solving this equation forces $\sqrt{-3}$ into our field. Thus, we may assert the following:

The matroid M is representable by the columns of a matrix over a field F iff F contains a root of the equation $x^2 - x + 1$.

2. Non-representable matroids

Given a matroid M and a field F, we are interested in determining whether M can be represented by the column vectors of a matrix with entries in the field F. More precisely, we define matroid representability as follows.

Definition 2.1. A matroid M is *representable over a field F* if there is a matrix N with entries taken from F so that:

- There is a bijection between the points of M and the columns of N, and
- A subset of points of M is independent if and only if the corresponding columns of N are linearly independent.

Question Q3 asks whether every matroid arises from a matrix, i.e., is every matroid representable over some field? The answer is no, and the next example proves this using a classic theorem from finite geometry.

Example 2.2. The configuration pictured on the left in Figure 4 is called the *Pappus* configuration. Its discovery dates to Pappus of Alexandria (c. 320 A.D.). The configuration is constructed as follows:

- Start with two 3-point lines abc and def.
- Now form the point g as the intersection of the two lines ae and bd, i.e., $g = ae \cap bd$.
- Continue to form the points $h = af \cap cd$ and $i = bf \cap ce$.
- Then the three points g, h and i must be collinear.

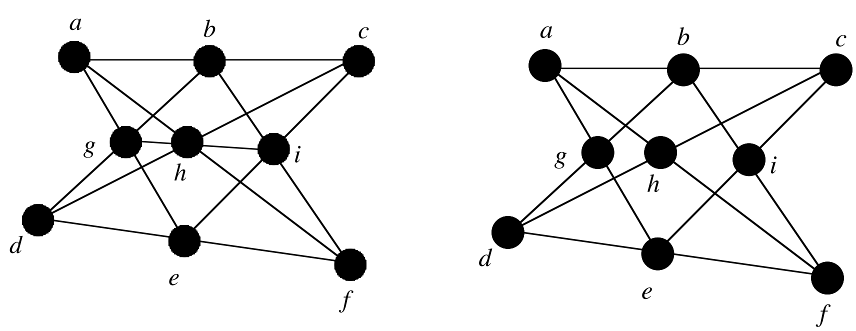

FIGURE 4. Left: The Pappus configuration. Right: The non-Pappus matroid. The points g, h and i must be collinear if M is representable over a field.

Now let M be the matroid depicted on the right of Fig. 4. This matroid is called the *non-Pappus* matroid, and g, h and i are not collinear in M. This is the key to next result.

Theorem 2.3. *The non-Pappus matroid is not representable over any field.*

Proof sketch. As in Example 1.5, we attempt to find coordinates for the 9 points of M. This gives the matrix:

$$A = \begin{array}{c} \begin{matrix} a & c & d & f & h & b & e & g & i \end{matrix} \\ \left[\begin{matrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & x & 1 & x & z \\ 0 & 0 & 1 & 1 & 1 & 0 & y & xy & y \end{matrix}\right] \end{array},$$

where x and y are indeterminates and z will be determined. Note the order we list the columns - this ordering will make our argument a little easier computationally. We first comment on the xy term appearing in the last coordinate of column g. First, since g is on the line bd, we know g must have first two coordinates in the ratio $1 : x$, so $g = [1, x, w]^T$, and we still need to determine w. Since g is also on the line ae, there is some linear combination of columns a and e that produce column g:

$$\alpha[1,0,0]^T + \beta[1,1,y]^T = [1,x,w]^T.$$

We get $\alpha = 1 - x$ and $\beta = x$; this gives $w = xy$.

For column i, a similar argument gives $i = [1,z,y]^T$, and z is determined from the linear combination involving bf:

$$\alpha[1,1,1]^T + \beta[1,x,0]^T = [1,z,y]^T.$$

Choosing $\alpha = y$ and $\beta = 1 - y$ produces $z = x + y - yx$. Note: we are being careful about always multiplying on the left; this will be important in the next paragraph.

Now the points g, h and i will be linearly dependent precisely when the determinant of the 3×3 submatrix formed by these three columns is 0.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & x & x+y-yx \\ 1 & xy & y \end{vmatrix} = 0.$$

But this determinant is $xy - yx$. Thus, the points g, h and i are collinear precisely when $xy = yx$. But we are working over a field (where multiplication is commutative), so this contradiction completes the proof.

Exercise 2. (Ingleton [5]) The proof of Theorem 2.3 shows M is representable over a division ring with two non-commuting indeterminates. Let M' be the matroid obtained from M by adding the line beh to the lines of M. Show M' is not representable over any field or division ring. (Hint: This additional dependence forces $x = y$.)

3. Characteristic sets

Whether the matroid M is representable over a field F often depends on F, or the field characteristic $\chi(F)$. We begin this section with two important examples, the *Fano* and *non-Fano* matroids.

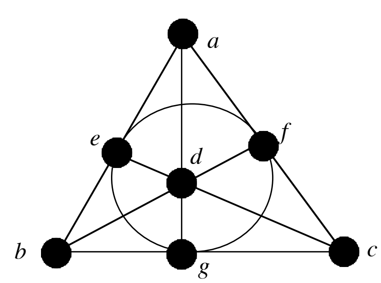

Figure 5. The Fano plane.

Example 3.1. Let F_7 be the matroid pictured in Fig. 5. This matroid is the *Fano plane*, and Proposition 1.3 tells us this is indeed a matroid - no two lines meet in more than one point. In fact, every pair of lines meet in exactly one point, where the line efg is represented in the figure by a circle. The Fano plane has lots of very nice properties. For instance, every point is on exactly three 3-point lines and every line has exactly three points. This point-line symmetry is a characteristic of *projective planes*; in fact, the Fano plane is the projective geometry $PG(2,2)$. We represent M with the following matrix.

$$\begin{array}{c} \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Now the entries in this matrix are *forced* in the same way the entries in previous examples were forced. Thus, the points e, f and g will be dependent if and only if the 3×3 determinant formed by these columns is 0. But

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2.$$

Thus, efg are collinear precisely when $2 = 0$, i.e., $\chi(F) = 2$.

Note that we can remove the line efg to form a new matroid F_7', the *non-Fano matroid* pictured in Fig. 6. This matroid is representable over fields where $2 \neq 0$, i.e., fields with $\chi(F) \neq 2$.

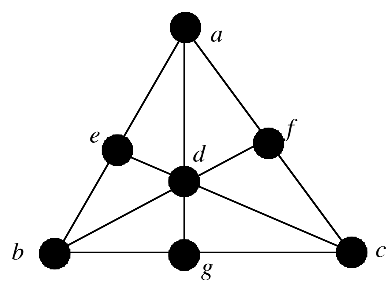

FIGURE 6. The non-Fano plane.

We summarize with a theorem.

Theorem 3.2. *Let F_7 be the Fano plane and F_7' be the non-Fano planes. Then F_7 is representable over a field if and only if $\chi(F) = 2$, and F_7' is representable over F if and only if $\chi(F) \neq 2$.*

This example motivates the next definition.

Definition 3.3. The *characteristic set* $\chi(M)$ of a matroid M is the subset of field characteristics over which M is representable.

Note $\chi(M) \subseteq P := \{0, 2, 3, 5, 7, 11, \dots\}$ for any matroid M. Theorem 3.2 implies $\chi(F_7) = \{2\}$ and $\chi(F_7') = P - \{2\}$. Theorem 2.3 gives $\chi(M) = \emptyset$ for the non-Pappus matroid M.

Exercise 3. (1) Show the uniform matroid of rank r on n points $U_{r,n}$ is representable over all characteristics, i.e., $\chi(U_{r,n}) = P$.

(2) Let M be the matroid of Example 1.5. Show $\chi(M) = P$.

An important question concerning characteristic sets was answered in the 1970's and early 1980's.

Q4. Which subsets of P can occur as the characteristic set of a matroid?

Representations over characteristic 0 play an important role in answering this question.

Theorem 3.4. *Let $\chi(M)$ be the characteristic set of the matroid M.*

(1) *If $0 \in \chi(M)$, then M is representable over all sufficiently large characteristics, i.e., $\chi(M)$ is cofinite.*
(2) *If $\chi(M)$ is infinite, then $0 \in \chi(M)$.*

Part (a) of Theorem 3.4 was proven in 1957 by Rado [8] using the Nullstellensatz. Part (b) was first established by Vamos [13] in 1970. It follows (almost) immediately from the compactness theorem in mathematical logic.

Putting parts (a) and (b) together gives the following corollary:

Corollary 3.5. *The characteristic set of a matroid M is either finite or cofinite.*

It turns out every finite subset of characteristics (necessarily excluding 0) can occur as the characteristic set of some matroid (Kahn [6]), and the same is true for cofinite subsets (Reid [9]). Kahn constructs examples using classical tools in projective geometry, especially cross-ratios. Reid uses geometric techniques to create algebraic conditions on the elements of the field, the *von Staudt calculus.*

4. Matroids representable over a given field

Rather than concentrating on the characteristic of the field a matroid M is represented over, it is often of interest to focus on the specific field. For instance, matroids representable over the 2-element field $GF(2)$ are especially easy to coordinatize; since every entry is 0 or 1, these matroids are *uniquely* representable. It turns out the same thing is true for $GF(3)$, the three element field. A matroid representable over $GF(2)$ is called *binary*; $GF(3)$-representable matroids are *ternary.* A matroid representable over all fields is *regular* or *unimodular*. (A matrix is *unimodular* if every subdeterminant is $0, 1$ or -1. Unimodular matroids can always be represented by such matrices.)

Exercise 4. (1) Show the uniform matroid $U_{2,n}$ is representable over a field F if and only if $|F| > n - 2$. Thus, $U_{2,3}$ is regular, $U_{2,4}$ is representable over all fields except $GF(2)$, and so on.
(2) Show the matroid M from Example 1.5 is representable over a field F if and only if $z^2 = -3$ has a solution in the field F. For prime fields, this is true precisely when $p = 3$ or $p \equiv 1 \pmod 3$.

A matroid N is a minor of a matroid M if you can obtain an isomorphic copy of N by deleting and contracting points of M. There are many characterizations of binary matroids - a rather impressive list of such characterizations appears in Oxley's text [7] in the chapter on binary matroids. The most important such characterization is the following, due to Tutte [12]:

Theorem 4.1. *A matroid is binary if and only if it has no minor isomorphic to the 4-point line $U_{2,4}$.*

Exercise 5. We have shown the following matroids are not binary: the matroid M in Example 1.5, the non-Fano plane and the non-Pappus configuration. For each of these matroids, find a $U_{2,4}$ minor.

For ternary matroids, the list of excluded minors is also known. The following theorem was first proven by Reid in the early 1970's, but the first published proofs were [1] and [11].

Theorem 4.2. *A matroid is ternary if and only if it has no minor isomorphic to one of the following 4 matroids:* $F_7, F_7^*, U_{2,5}, U_{3,5}$.

Exercise 6. Show the non-Pappus matroid is not ternary by finding one of the excluded minors listed in Theorem 4.2.

Several important questions are related to representations over specific fields.

- Q5. If M is binary, then what other fields (with characteristic $\neq 2$) might M be representable over?
- Q6. If M is ternary, then what other fields might M be representable over?
- Q7. (Rota's conjecture - 1971) Can the class of matroids representable over a given finite field F be characterized by a finite list of minimal *excluded* or *forbidden* minors?
- Q8. (Infinite antichains) Related to Rota's conjecture, but logically distinct: If $\{M_1, M_2, \dots\}$ is an infinite collection of matroids all representable over a finite field F, then must some M_i be a minor of some other M_j?

Q5 was answered by Tutte [12].

Theorem 4.3. *Suppose M is binary. Then either M is representable over all fields (i.e., M is regular), or $\chi(M) = \{2\}$, i.e., M is representable only over fields of characteristic 2.*

Since regular matroids can be represented by totally unimodular matrices, i.e., matrices all of whose subdeterminants are 0, 1 or -1, Tutte's theorem can be thought of as a characterization of regular matroids.

Whittle [14, 15] resolved Q6 in 1995. We state the characteristic set implications of his results.

Theorem 4.4. *Suppose M is ternary. Then exactly one of the following holds:*

(1) $\chi(M) = P$.
(2) $\chi(M) = P - \{2\}$.
(3) $\chi(M) = \{3\}$.

In fact, Whittle proved much more. For $q = 2, 3, 4, 5, 7$ and 8, he characterized the class of matroids representable over $GF(3)$ and $GF(q)$. For $q = 2$, this is the class of regular matroids; this follows from Theorem 4.3. Whittle introduced the classes of $\sqrt[6]{1}$-matroids, dyadic matroids, and near-regular matroids, all of which involve representing M by matrices whose determinants are restricted to a small number of values. Then these classes can be combined in various ways to produce the desired representation classes.

Rota's conjecture Q7 remains open. The entire collection of excluded minors for representation over $GF(q)$ is only known for $q = 2, 3$ and 4 (the minors for $GF(4)$ were obtained by Geelen, Gerards and Kapor in [3]). In fact, if $Ex(q)$ is the set of (minimal) excluded minors for representation over $GF(q)$, then it is not currently known if $Ex(q)$ is finite for any $q \geq 5$.

Question Q8 is intriguing because it is closely related to the very successful graph minors project of Robertson and Seymour [10]. Among many other important results, they prove Wagner's long standing conjecture that, among any infinite collection of graphs, one of the graphs must be a minor of another. Thus, Q8 is true for graphic matroids - there is no infinite antichain in the partially ordered set of all graphs (partially ordered by minor relation). Important and deep extensions of this work to binary matroids appear in [4].

Acknowledgement We thank Liz McMahon for her helpful comments on a draft of this manuscript.

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