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Series–parallel posets and the Tutte polynomial

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Abstract

We investigate the Tutte polynomial $f(P; t, z)$ of a series–parallel partially ordered set P. We show that $f(P)$ can be computed in polynomial-time when P is series–parallel and that series–parallel posets having isomorphic deletions and contractions are themselves isomorphic. A formula for $f(P^*)$ in terms of $f(P)$ is obtained and shows these two polynomials factor over $\mathbf{Z}[t, z]$ the same way. We examine several subclasses of the class of series–parallel posets, proving that $f(P) \neq f(Q)$ for non-isomorphic posets P and Q in the largest of these classes. We also give excluded subposet characterizations of the various subclasses.

Keywords: Tutte polynomial; Series–parallel poset

0. Introduction

The Tutte polynomial is a two-variable which has been defined and studied in depth for graphs and matroids. An extensive introduction to the theory can be found in [2]. Recently, this definition has been extended to greedoids [7], and examined in detail [6] for partially ordered sets (posets), which form a class of greedoids. This paper continues the study begun in [6], concentrating especially on series–parallel posets and various subclasses of series–parallel posets.

Series–parallel posets (denoted SP posets) form an attractive class of posets because their recursive structure permits many polynomial-time algorithms. For example, although scheduling problems are NP-complete for arbitrary partial orders, efficient algorithms exist if the partial order is an SP poset. Further, Valdes et al. [14] show there is a linear-time algorithm for recognizing whether a given poset belongs to the class SP. Many other authors have considered SP posets for a variety of purposes. For example, Stanley [12] uses Pólya's Theorem to get a generating function for the number of SP posets on n elements. In a different direction, let $N(P)$ be the number of

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order ideals in P and let $N(x)$ be the number of order ideals in P which contain x. Then Faigle et al. [5] show that searching in SP posets for an element x with $\frac{1}{4} \leqslant N(x)/N(P) \leqslant \frac{3}{4}$ (the best possible bound) can be done efficiently, while Provan and Ball [10] show that even determining $N(P)$ is #P-complete for an arbitrary poset.

In this paper, we show that SP posets are well-behaved with respect to the Tutte polynomial in several respects. In Section 1, we show that computing the polynomial $f(P)$ can be done in polynomial-time for an SP poset P, but the problem of computing the simple evaluation of $f(P)$ at $t = 1$, $z = 0$ is #P-complete (in the sense of Valiant [15]) for an arbitrary poset P (Proposition 3). The other main result of that section (Theorem 8) shows that two SP posets P and Q with $P/x \cong Q/y$ and $P - I^*(x) \cong Q - I^*(y)$ must have $P \cong Q$. (P/x and $P - I^*(x)$ correspond to contraction and deletion, familiar operations in matroid and greedoid theory but not in poset theory.)

In Section 2, we derive a formula relating $f(P)$ to $f(P^*)$, where P^* is the dual of P. As an application of the formula, we immediately get that $f(P)$ is an irreducible polynomial precisely when $f(P^*)$ is. We then consider several subclasses of SP posets, with the main result (Theorem 15) that, for one of these classes, $f(P)$ is a complete invariant in the sense that non-isomorphic posets P and Q in the class have $f(P) \neq f(Q)$. We also give (Proposition 17) excluded subposet characterizations of each of the classes.

We now recall a few definitions we will need. See Stanley [13] or Rival [11] for more details. Let P be a poset and let $I \subseteq P$. Then I is an *order ideal* if whenever $x \in I$ and $y < x$, then $y \in I$. For $S \subseteq P$, define the *rank of* S, denoted $r_p(S)$ (or simply $r(S)$) as follows:

$$r_p(S) \equiv \max_{I \subseteq S} \{ |I| : I \text{ is an order ideal} \}.$$

The *Tutte polynomial* of P is defined by

$$f(P; t, z) \equiv \sum_{S \subseteq P} t^{|P| - r(S)} z^{|S| - r(S)}$$

The definition of rank comes directly from greedoid theory. See [1] for details. We now recall several elementary poset operations.

Direct sum: $P + Q$ is a poset on $P \cup Q$ with $x \leqslant y$ in $P + Q$ if either
(a) $x, y \in P$ and $x \leqslant y$ in P or
(b) $x, y \in Q$ and $x \leqslant y$ in Q.

Ordinal sum: $P \oplus Q$ is a poset on $P \cup Q$ with $x \leqslant y$ in $P \oplus Q$ if either
(a) $x, y \in P$ and $x \leqslant y$ in P or
(b) $x, y \in Q$ and $x \leqslant y$ in Q or
(c) $x \in P$ and $y \in Q$.

Ordinal product: $P \otimes Q$ is a poset on $\{(x, y): x \in P \text{ and } y \in Q\}$ with $(x, y) \leqslant (x', y')$ in $P \otimes Q$ if
(a) $x = x'$ in P and $y \leqslant y'$ in Q or
(b) $x < x'$ in P.

We let **1** be the one-element poset. A poset is a *series–parallel* poset if it can be built up recursively from **1** by using the operations of direct sum and ordinal sum. The Hasse diagrams of the posets formed by the above operations can all be obtained from the Hasse diagrams of P and Q by straightforward techniques. The reader can consult [13] or work out the details directly. Finally, the *dual* P^* of a poset P is obtained by flipping the Hasse diagram of P, i.e., $x \leqslant y$ in P^* iff $y \leqslant x$ in P.

If A is an antichain in a poset P, let $I(A)$ and $I^*(A)$ be the order ideal and order filter, respectively, generated by A. Further, we let $\overline{I(A)}$ and $\overline{I^*(A)}$ be the ideal and filter strictly generated by A, respectively. Thus

$$I(A) = \{x \in P\colon x \leqslant y \text{ for some } y \in A\},$$

$$I^*(A) = \{x \in P\colon x \geqslant y \text{ for some } y \in A\},$$

$$\overline{I(A)} = \{x \in P\colon x < y \text{ for some } y \in A\}$$

and

$$\overline{I^*(A)} = \{x \in P\colon x > y \text{ for some } y \in A\}.$$

We remark that I can be regarded as a bijection between the set of all antichains of P (which we denote $A(P)$) and the set of all order ideals. I^* gives a bijection between $A(P)$ and set of all order filters. For ease of notation, we will write $I(x)$, $I^*(x)$, etc. for $I(\{x\})$, $I^*(\{x\})$, etc. when $A = \{x\}$. The next proposition is proven in [6].

Proposition A (Proposition 2.3 [6]). *Let P be a poset and P^* be the dual of P. Then*

$$\text{(a) } f(P;t,z) = \sum_{A \in A(P)} t^{|I^*(A)|}(z+1)^{|\overline{I^*(A)}|}$$

and

$$\text{(b) } f(P^*;t,z) = \sum_{A \in A(P)} t^{|I(A)|}(z+1)^{|\overline{I(A)}|}.$$

(In part (b), $I(A)$ and $\overline{I(A)}$ are computed in P, not P^*.)

Thus, we can interpret the polynomial $f(P)$ as a generating function for the number of ordered pairs $(|I^*(A)|, |\overline{I^*(A)}|)$ for all antichains A in the poset. For the rest of this paper, we will set $y = (z+1)$ in $f(P)$ to simplify notation.

One of the most useful features of the Tutte polynomial of a greedoid is the recursive deletion–contraction formula (Proposition 3.2 of [7]) it satisfies. Another result we will need concerns the application of this formula to $f(P)$. If x is minimal in P, then define P/x to be the poset on the set $P - \{x\}$ with the inherited partial order, i.e., the induced subposet on $P - \{x\}$. Similarly, define the poset $P - I^*(x)$ to be simply the induced subposet on the set $P - I^*(x)$. (In terms of the greedoid $G(P)$

associated to the poset P, P/x corresponds to the contraction in $G(P)$ and $P - I^*(x)$ essentially corresponds to deletion. See [6] for more details.)

Proposition B (Proposition 2.4 and Corollary 2.6 in [6]). *Let x be minimal in a poset P. Then*

(a) $f(P;t,y) = f(P/x;t,y) + t^{|I^*(x)|}y^{|I^*(x)|-1}f(P - I^*(x);t,y)$

(b) $f(P^*;t,y\} = (ty)f((P/x)^*;t,y) + (1 + t - ty)f((P - I^*(x))^*;t,y)$.

1. Series–parallel posets

We begin by recalling the behavior of the polynomial under direct sum and ordinal sum.

Proposition 1 (Proposition 4.1 [6]). *Let P and Q be two posets.*

(a) *Direct sum*: $f(P + Q) = f(P)f(Q)$;

(b) *Ordinal sum*: $f(P \oplus Q) = f(Q) + (ty)^{|Q|}(f(P) - 1)$;

Example 2. Consider the poset P of Fig. 1. The reader can check that $f(P) = (t+1)^2 + 2t^2(t+1)y + t^4y^2 + t^4(t+1)y^3$. We claim that there is no SP poset Q with $f(P) = f(Q)$. To see this, we note that such a poset Q must have five elements, 2 which are maximal and 2 which are minimal. Furthermore, $f(P)$ is irreducible over $\mathbf{Z}[t, y]$, which rules out SP posets which are direct sums. The only SP poset which meets these requirements is the poset Q shown in Fig. 1. But Q has only two 2-element antichains and P has five 2-element antichains, so $f(P) \neq f(Q)$.

Several authors (e.g., [4, 5, 8, 10]) have recently considered questions of computational complexity with respect to computing the Tutte polynomial and various evaluations of the Tutte polynomial for graphs and certain classes of matroids. For example, Colbourn et al. [4] show that when M is transversal matroid, computing the Tutte polynomial $T(M;t,z)$ at the point (a,b) in the (t,z) plane (in which a and b are algebraic numbers) is #P-complete unless $ab = 1$, in which case it is polynomial-time computable. Proposition 3(a) shows that the computation of $f(P)$ is polynomial-time for an SP poset P, while Proposition 3(b) shows that it is unlikely that an efficient algorithm exists for computing $f(P)$ for an arbitrary poset P.

Fig. 1.

Proposition 3. (a) *If P is a series–parallel poset, then the polynomial $f(P; t, y)$ can be computed in polynomial-time.*

(b) *Evaluating $f(P; t, y)$ at $t = 1$, $y = 1$ is #P-complete for a general poset P.*

Proof. (a) As in the proof of Theorem 3 of [5], we first find a decomposition tree for P in polynomial-time using the recognition algorithm of Valdes et al. [14]. (A *decomposition tree* is a binary tree in which the internal nodes are labeled either S or P, corresponding to the operations of either ordinal sum (series) or direct sum (parallel), and the external nodes are labeled by the elements of the poset P. Clearly, such a tree completely determines P.) Using the decomposition and the recursive formulas from Proposition 1, we can easily compute $f(P; t, y)$ in polynomial-time.

(b) From Proposition A(a), we have $f(P; t, y)$ evaluated at $t = 1$, $y = 1$ equals the number of order filters in P. From Provan and Ball [10], the problem of computing this invariant is #P-complete. □

We can view the computation of $f(P)$ from the decomposition tree as a recursive characterization of the class of polynomials $f(t, y)$ which can occur as the polynomial of an SP poset. We omit the intermediate proof of the next result.

Proposition 4. *Let $F \subseteq Z[t, y]$ be defined recursively by:*

(i) $(t + 1) \in F$;

(ii) $f, g \in F \Rightarrow fg \in F$;

(iii) $f, g \in F \Rightarrow f + (ty)^n (g - 1) \in F$, *where n is the t-degree of f;*

Then F is precisely the set of polynomials which can occur as $f(P)$ for an SP poset P.

From Proposition 1, it is easy to compute $f(P + Q)$ or $f(P \oplus Q)$ from the polynomials $f(P)$ and $f(Q)$. It is natural to ask if it is possible to reverse either of these operations, i.e., can we determine $f(P)$ and $f(Q)$ from either $f(P + Q)$ or $f(P \oplus Q)$? We say a poset P is *direct sum irreducible* if it cannot be written as the direct sum of two non-empty posets. Similarly, P is *ordinally irreducible* if it cannot be written as the ordinal sum of two non-empty posets. Example 3.1 of [6] shows that

$f($  $) = f($  $)$

so it is not possible to determine $f(P)$ and $f(Q)$ from $f(P + Q)$, even when P and Q are direct sum irreducible. (Of course, given $f(P + Q)$, it is always possible to find some posets P' and Q' such that $f(P' + Q') = f(P + Q)$ by simply factoring $f(P + Q)$ over $\mathbf{Z}[t, y]$ and then finding appropriate posets P' and Q' by exhaustive search.) In contrast with this example, Proposition 6 shows that it is possible to determine $f(P)$ and $f(Q)$ from $f(P \oplus Q)$ when P and Q are ordinally irreducible. We begin with a lemma.

Lemma 5. *A poset R is an ordinal sum of two non-empty posets if and only if there is some positive integer $k < |R|$ such that there is exactly one order ideal of size k in R.*

Proof. First suppose $R = P \oplus Q$ for some non-empty posets P and Q. Then there is only one order ideal in R of size $|P|$, since any order ideal of $P \oplus Q$ which includes some element of Q must include all of P. Conversely, if there is only one order ideal I of size k for some $0 < k < |R|$, then we claim $R \cong I \oplus (R - I)$, where $R - I$ represents the subposet induced by the complement of I in R. To show this, it suffices to show that every element of I is less than every element of $R - I$, where all comparisons are made in R. If this were not true, then there must be an incomparable pair (x, y) with $x \in I$ and $y \in R - I$, and we may take x to be maximal in I and y to be minimal in $R - I$. Then the set $I \cup \{y\} - \{x\}$ is also an order ideal in R of size k, contradicting the assumption. □

Proposition 6. *Suppose P and Q are ordinally irreducible posets. If $f(R) = f(P \oplus Q)$ for some poset R, then $R = P' \oplus Q'$ where $f(P) = f(P')$ and $f(Q) = f(Q')$.*

Proof. Expand the evaluation of $f(R; t, y)$ at $y = 1$ as a polynomial in t:

$$a_{|R|}t^{|R|} + a_{|R|-1}t^{|R|-1} + \cdots + a_1 t + 1.$$

By Proposition $A(a)$, a_i is the number of order ideals in R of size $|R| - i$. From Lemma 5 and the fact that $f(R) = f(P \oplus Q)$, we get $a_{|Q|} = 1$. Furthermore, the ordinal irreducibility of P and Q implies $a_{|Q|}$ is the unique coefficient equal to 1 (whose index is strictly between 0 and $|R|$) in $f(R; t, 1)$, i.e., if $a_k = 1$ for some $0 < k < |R|$, then $k = |Q|$. Applying the lemma again, we find that $R = P' \oplus Q'$ for some posets P' and Q' and $|Q'| = |Q|$. From Proposition 1, we now have

$$f(R) = f(Q) + (ty)^{|Q|}(f(P) - 1) = f(Q') + (ty)^{|Q'|}(f(P') - 1).$$

Since $|Q'| = |Q|$ and the term of highest t-degree appearing in both $f(Q)$ and $f(Q')$ is $t^{|Q|}$, and the term of lowest t-degree appearing in both $(ty)^{|Q|}(f(P) - 1)$ and $(ty)^{|Q'|}(f(P') - 1)$ is $t^{|Q|+1}$, we get $f(P) = f(P')$ and $f(Q) = f(Q')$. □

We can generalize Proposition 6 as follows. Suppose $f(P) = f(P_1 \oplus P_2 \oplus \cdots \oplus P_n)$ where each P_i is ordinally irreducible. Then $P = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$, where $f(P_i) = f(Q_i)$ for each i. Thus, we can uniquely determine $f(P_i)$ for each i from $f(P)$.

In Example 3.1 of [6], one poset is an SP poset and the other is not. (See the comments preceding Lemma 5.) Hence it is possible for $f(P) \in F$ with P not an SP poset. Thus, $f(P)$ does not distinguish the class of SP posets among the class of all posets.

We will need the next characterization of SP posets, which is well known.

Theorem 7 (Theorem 1 [14]). *P is an SP poset if and only if P has no induced subposet isomorphic to N.*

Proposition B can be used in the following way to construct non-isomorphic posets P and Q with $f(P) = f(Q)$. Beginning with a poset R, we create posets P and Q by adding a new minimal element x or y, respectively, to R in such a way that $P/x = R = Q/y$ and also $P - I^*(x) \cong Q - I^*(y)$. In fact, all known (minimal) such pairs P and Q are formed this way. The next theorem shows that this technique will not produce such a pair when both posets are SP.

Theorem 8. *Suppose P and Q are SP posets and, for some minimal $x \in P$ and $y \in Q$, $P - I^*(x) \equiv Q - I^*(y)$ and $P/x \cong Q/y$. Then $P \cong Q$.*

Proof. We proceed by contradiction, supposing that the pair (P, Q) is a minimum size counterexample, i.e., with $|P| = |Q|$ as small as possible.

Case 1: P is direct sum irreducible. Then $P = P_1 \oplus P_2$ for some SP posets P_1 and P_2, where we may assume P_1 is ordinally irreducible. Since $f(P) = f(Q)$, it follows from Proposition 4 that $Q = Q_1 \oplus Q_2$ for some SP posets Q_1 and Q_2, $|P_i| = |Q_i|$ and Q_1 is also ordinally irreducible.

Now $P_1/x \oplus P_2 = P/x \cong Q/y = Q_1/y \oplus Q_2$, so $P_1/x \cong Q_1/y$ and $P_2 \cong Q_2$. Similarly, $P_1 - I^*(x) = P - I^*(x) \cong Q - I^*(y) = Q_1 - I^*(y)$, so $P_1 - I^*(x) \cong Q_1 - I^*(y)$. Since P and Q were chosen to be a minimal counterexample, we must have $P_1 \cong Q_1$. Thus, $P \cong Q$, which completes this case.

Case 2: P can be written as a direct sum of smaller posets. Then Proposition 4 and the facts that $f(P) = f(Q)$ and $P = P_1 + \cdots + P_k$ with $k \geqslant 2$ (and each P_i is direct sum irreducible) together imply that $Q = Q_1 + \cdots + Q_n$ with $n \geqslant 2$ (and each Q_j is direct sum irreducible). (Otherwise, Q is the ordinal sum of two SP posets, which, by Proposition 4, forces P to be an ordinal sum.) By minimality of the counterexample, we may assume no P_i is isomorphic to any Q_j for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$. We also order these posets so that $x \in P_1$ and $y \in Q_1$.

Since $P/x \cong Q/y$, $P/x = P_1/x + P_2 + \cdots + P_k$ and $Q/y = Q_1/y + Q_2 + \cdots + Q_n$, we must have $P_1/x \cong Q_2 + \cdots + Q_n + R$ and $Q_1/y \cong P_2 + \cdots + P_k + R$ for some SP poset R. Similarly, we also have $P_1 - I^*(x) \cong Q_2 + \cdots + Q_n + S$ and $Q_1 - I^*(y) \cong P_2 + \cdots + P_k + S$ for some SP poset S. Furthermore, $R \neq \emptyset$, since $|P_1/x| = |R| + \sum_{j=2}^{n} |Q_j|$, $|P_1 - I^*(x)| = |S| + \sum_{j=2}^{n} |Q_j|$ and $|P_1/x| > |P_1 - I^*(x)|$. Now x cannot be the unique minimal element in P_1, since this would force $P_1 - I^*(x) = \emptyset$. Thus there is some element $z \in P_1$ such that x and z are incomparable. We may assume z is in the component of P_1/x which is isomorphic to Q_2 and that z is minimal in P_1 (or else we could replace z by any minimal element which it is greater than). Since Q_2 cannot be written as a direct sum, we must have $Q_2 = A \oplus B$, so z corresponds to an element of A. Let u correspond to an element of $B(\neq \emptyset)$, so $x < u$ and $z < u$. Finally, let $v > x$ be any element of P_1 which is in the component of P_1/x isomorphic to R ($\neq \emptyset$). Note that v is incomparable with u and with z since it is not in the component of P_1/x which is isomorphic to Q_2. Then in P_1, we have the following inequalities: $x < u$, $x < v$, $z < u$ and (x, z), (u, v) and (z, v) all form incomparable pairs. Thus these four elements form N as an induced subposet, which contradicts the fact that P is an SP poset (Theorem 7). This completes the proof. □

Our final result in this section is also a negative result on constructing non-isomorphic SP posets P and Q with $f(P) = f(Q)$. We omit the proof.

Proposition 9. *If* $f((A \oplus B) + (C \oplus D)) = f((A \oplus D) + (C \oplus B))$ *if and only if* $f(A) = f(C)$ *or* $f(B) = f(D)$.

2. Duality and subclasses of series–parallel posets

Theorem 4.5 of [6] shows that $f(P^*)$ can be determined from $f(P)$, but does not give an algebraic connection between these two polynomials. We give such a formula now; we will need (a corollary of) this result when we examine subclasses of SP posets below.

Theorem 10. *If P is a poset with n elements, then*

$$f(P^*; t, y) = (ty)^n f\left(P; \frac{1 + t - ty}{ty}, \frac{1}{1 + t - ty}\right).$$

Proof. For convenience, set $a = (1 + t - ty)/(ty)$ and set $b = (1 + t - ty)^{-1}$. We use induction on n. If $n = 1$, then $f(P) = f(P^*) = t + 1$. From the formula, we get $f(P^*) = (ty)(a + 1) = t + 1$, so the formula holds.

Now assume the formula is valid for all posets on $n - 1$ elements for $n \geqslant 2$. By proposition $B(b)$, we have

$$f(P^*; t, y) = (ty) f((P/x)^*; t, y) + (1 + t - ty) f((P - I^*(x))^*; t, y).$$

By induction, we can write

$$f((P/x)^*; t, y) = (ty)^{n-1} f(P/x; a, b)$$

and

$$f((P - I^*(x))^*; t, y) = (ty)^{n - |I^*(x)|} f(P - I^*(x); a, b).$$

Thus,

$$f(P^*; t, y) = (ty)^n f(P/x; a, b) + (ty)^n (ty)^{-|I^*(x)|} b^{-1} f(P - I^*(x); a, b).$$

Now $ab = (ty)^{-1}$, so the last equation can be written as

$$f(P^*; t, y) = (ty)^n \{ f(P/x; a, b) + a^{|I^*(x)|} b^{|I^*(x)| - 1} f(P - I^*(x); a, b) \}.$$

By proposition $B(a)$, the right hand side of the last equation is simply $(ty)^n f(P; a, b)$, so we are done. $\square$

It is easy to see that the formula in Theorem 10 is consistent with the involution property of duality ($P^{**} = P$). Applying this formula to $f(P^{**}, t, y)$ gives

$f(P^{**};t,y) = (ty)^n f(P^*;a,b)$, where $a = (1+t-ty)/(ty)$ and $b = (1+t-ty)^{-1}$ as in the proof of the theorem. Applying the formula to $f(P^*)$ gives

$$f(P^{**};t,y) = (ty)^n (ab)^n f\left(P; \frac{1+a-ab}{ab}, \frac{1}{1+a-ab}\right).$$

But it is easy to see that $t = (1+a-ab)/(ab)$, $y = (1+a-ab)^{-1}$ and $ab = (ty)^{-1}$, so this reduces to $f(P^{**};t,y) = f(P;t,y)$.

We also remark that a non-inductive proof of Theorem 10 can be constructed by using Proposition A and the formula

$$\sum_{k \geq 0} u(i,k)\binom{k}{j} = \sum_{k \geq 0} v(i-j,k)\binom{k}{j},$$

where $u(i,j)$ counts the number of elements of rank i which cover exactly j elements in the distributive lattice $J(P)$ of order ideals of P and $v(i,j)$ counts the number of elements of rank i of $J(P)$ which are covered by exactly j elements. (This formula appears as problem 21 on page 157 of [13].)

Corollary 11. *$f(P;t,y)$ is irreducible over $\mathbf{Z}[t,y]$ if and only if $f(P^*;t,y)$ is irreducible.*

Proof. If $f(P) = g(t,y)h(t,y)$, then each non-zero term mt^iy^j appearing in either factor must have $i > j$ (unless $i = j = 0$). (Otherwise, we could find the terms in which $j - i$ is maximized in g and in h and multiply these two terms together to create a non-constant term in f with t-exponent $\leqslant$ y-exponent, which is a contradiction). Thus, the induced factorization of $f(P^*)$ is also a factorization over $\mathbf{Z}[t,y]$. □

We now turn our attention to subclasses of SP posets. The class of SP posets satisfies each of the closure properties listed below. We now define several subclasses by selecting various subsets of these properties under which the subclass will be closed. Let A_n denote an n-element antichain and let K_i denote an (as yet unspecified) class of posets. Then define properties P_j and P_j^* $(0 \leqslant j \leqslant 3)$ which the class K_i may or may not enjoy as follows:

P_0: $\mathbf{1} \in K_i$

P_1: If $P, Q \in K_i$, then $P \oplus Q \in K_i$ (closure under direct sum)

P_2: If $P \in K_i$, then $P \oplus \mathbf{1} \in K_i$ (closure under 'capping')

P_2^*: If $P \in K_i$, then $\mathbf{1} \oplus P \in K_i$ (closure under 'cupping')

P_3: If $P \in K_i$, then $P \oplus A_n \in K_i$ (closure under 'multi-capping')

P_3^*: If $P \in K_i$, then $A_n \oplus P \in K_i$ (closure under 'multi-cupping')

Now define the following subclasses of SP:

K_1 satisfies P_0, P_1 and P_2;

K_1^* satisfies P_0, P_1 and P_2^*;

K_2 satisfies P_0, P_1, P_2 and P_2^*;

K_3 satisfies P_0, P_1, P_2 and P_3^*;

K_3^* satisfies P_0, P_1, P_2^* and P_3;

K_4 satisfies P_0, P_1, P_3 and P_3^*.

There are many easy relationships among these classes. For example, $K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4$ and $P \in K_i$ if and only if $P^* \in K_i^*$ for $i = 1$ and 3. Furthermore, the classes K_1 and K_1^* can each be identified with the class of *rooted trees* in the following way. A *rooted tree* T is a tree with a distinguished vertex. We then associate a poset $B(T)$ to T in the following way. The elements of $B(T)$ are the edges of T, and for edges a and b of T, define $a < b$ in $B(T)$ if and only if the unique path in T from the root $*$ to the terminal vertex of b contains a. If we use the convention that the rooted tree is drawn with the root at the bottom and all edges are directed up as we move away from the root, then the Hasse diagram of the poset which represents $B(T)$ is obtained from T by simply erasing the root and all edges incident with the root. We illustrate the correspondence in Fig. 2. The order ideals in $B(T)$ correspond to the edge sets of rooted subtrees of T; the associated greedoid is called the *branching greedoid* of T. In the dual poset $B^*(T)$, order ideals correspond to the complements of rooted trees; the associated greedoid is called the *rooted pruning greedoid* of T. The branching greedoid can be defined on any rooted graph or rooted digraph (see [7, 9]), while the pruning greedoid can be defined on unrooted trees. The application of the Tutte polynomial to unrooted trees is explored in [3].) Clearly $\{B(T)$: T is a rooted tree$\} = K_1^*$ and $\{B^*(T)$: T is a rooted tree$\} = K_1$.

Theorem 2.8 of [7] shows that non-isomorphic members of K_1^* have distinct Tutte polynomials, while Theorem 1b of [3] proves the same result for non-isomorphic members of K_1. (A direct proof of the equivalence of these two results now follows from Theorem 10.) Theorem 3.10 of [6] extends this result to the class K_2. We will now extend this result once more to the class K_4. All four of these proofs require lemmas on the irreducibility of $f(P)$ when P is the poset resulting from the application of the various operations of capping, cupping and multi-cupping.

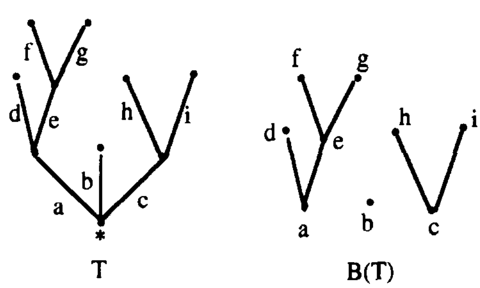

Fig. 2.

Lemma 12. *Let P be any poset with $x \in P$ such that $x < w$ for all non-minimal $w \in P$. Then $f(P; t, y) = (t+1)^k g(t, y)$ for some $0 \leqslant k \leqslant M$ where M is the number of maximal elements of P and g is an irreducible polynomial over $\mathbf{Z}[t, y]$.*

Proof. Write $f(P; t, y) = f_r(t)y^r + f_{r-1}(t)y^{r-1} + \cdots + f_0(t)$ for some $r \geqslant 0$ and suppose $f(P) = g(t, y)h(t, y)$, where all factoring takes place in the ring $\mathbf{Z}[t, y]$. Then, as in the proof of Corollary 11, each non-zero term $t^c y^d$ in $g(t, y)$ and $h(t, y)$ must have $c > d$ (or $c = d = 0$).

Now write $g(t, y) = g_a(t)y^a + g_{a-1}(t)y^{a-1} + \cdots + g_0(t)$ and $h(t, y) = h_b(t)y^b + h_{b-1}(t)y^{b-1} + \cdots + h_0(t)$, where $a + b = r$ and $a, b > 0$. By the above argument, t^{a+1} divides $g_a(t)$ and t^{b+1} divides $h_b(t)$ so, t^{r+2} divides $f_r(t)$. But $f_r(t) = t^{|P|} + \cdots + mt^{r+1}$ for some *positive* integer m, because the singleton antichain corresponding to the element x will contribute the term $t^{r+1}y^r$ to $f(P)$. This contradiction forces $a = 0$ or $b = 0$, i.e., $f(P) = g(t, y)h(t)$ and g is irreducible over $\mathbf{Z}[t, y]$.

To determine $h(t)$, note that $f(P; t, 0) = (t+1)^M$, where M is the number of maximal elements of P. Thus, $(t+1)^M = g(t, 0)h(t)$, so $h(t) = (t+1)^k$ for some $k \leqslant M$. $\square$

Lemma 13. *If $f(P \oplus Q; t, y) = g(t, y)h(t)$, then $h(t) = \pm 1$.*

Proof. From Proposition 1, $f(P \oplus Q) = f(Q) + (ty)^{|Q|}[f(P) - 1]$. Write $f(P \oplus Q) = f_r(t)y^r + f_{r-1}(t)y^{r-1} + \cdots + f_0(t)$ and note that $f_0(t) = (t+1)^M$ and $f_{|Q|}(t) = t^{|Q|}[(t+1)^N - 1]$, where M is the number of maximal elements of Q and N is the number of maximal elements of P. Thus, $f_0(t)$ and $f_{|Q|}(t)$ are relatively prime, so $h(t) = \pm 1$. $\square$

The next lemma follows immediately from the previous two.

Lemma 14. *$f(A_n \oplus P)$ is irreducible over $\mathbf{Z}[t, y]$ for any poset P.*

The next theorem, which generalizes Theorem 3.10 of [6], is the main result in this section.

Theorem 15. *If $P_1, P_2 \in K_4$, then $f(P_1) = f(P_2)$ if and only if P_1 and P_2 are isomorphic.*

Proof. We show by induction that the poset $P \in K_4$ can be uniquely reconstructed from $f(P)$; this is equivalent to the result. The result is trivial for $|P| = 1$. Assume we are given $f(P)$ for some poset $P \in K_4$ with $|P| > 1$.

Case 1: P is not a direct sum of smaller posets. (Note that this can be determined solely from $f(P)$ by the proof of Proposition 6.) Then $P = P_1 \oplus P_2$ for some non-empty posets P_1 and $P_2 \in K_4$. By the proof of Proposition 6, we can determine both $f(P_1)$ and $f(P_2)$ from $f(P)$. By induction, we then reconstruct the two posets P_1 and P_2 which allows us to uniquely reconstruct P.

Case 2: P is a direct sum of smaller posets. We write $P = P_1 + \cdots + P_k$ for some $k \geqslant 2$, where each $P_i \in K_4$ is direct sum irreducible. Now factor $f(P)$ into irreducibles over $\mathbf{Z}[t, y]$. By Corollary 11 and Lemma 14, $f(P_i)$ is irreducible over $\mathbf{Z}[t, y]$ for all i. By induction, we can then reconstruct each poset P_i, so we can reconstruct P and we are done. □

The next result also relates the factorization of $f(P)$ to the poset P, generalizing 3.9 of [6]. The proof follows immediately from Theorem 10 and Lemma 14.

Corollary 16. *Suppose P is a poset with m minimal elements and M maximal elements and that* $f(P) = f_1(t,y) f_2(t,y) \cdots f_n(t,y)$, *where each* $f_i(t, y)$ *is irreducible over* $\mathbf{Z}[t, y]$. *Then* $n \leqslant \min[m, M]$.

Let F_i (or F_i^*) $= \{f(P; t, y): P \in K_i$ (or K_i^*)$\}$. Then each F_i (or F_i^*) is a multiplicatively closed subset of $\mathbf{Z}[t, y]$ and it is possible to give recursive characterizations of each F_i (or F_i^*) as in Proposition 4. For example, Propositions 9 and 10 of [3] give such characterizations of the classes K_1 and K_1^*. We leave the rest of these characterizations (all of which follow from applying Proposition 1 to the subclass under consideration) to the interested reader and instead turn our attention to excluded induced subposet characterizations, as in Theorem 7. We conclude with the following proposition.

Proposition 17. *Let posets* P_i $(1 \leqslant i \leqslant 5)$ *be the posets of Fig.* 3.

(a) $P \in K_1$ *if and only if P has no induced subposet isomorphic to* P_1. *Dually,* $P \in K_1^*$ *if and only if P has no induced subposet isomorphic to* P_1^*.

(b) $P \in K_2$ *if and only if P has no induced subposet isomorphic to* P_2 *or* P_3.

(c) $P \in K_3$ *if and only if P has no induced subposet isomorphic* P_2 *or* P_4. *Dually,* $P \in K_3^*$ *if and only if P has no induced subposet isomorphic to* P_2 *or* P_4^*.

(d) $P \in K_4$ *if and only if P has no induced subposet isomorphic to* P_2 *or* P_5.

Proof. We prove *d*; the proofs for the rest are similar. Since the operations of direct sum, multi-capping and multi-cupping can never produce either P_2 or P_5, it is clear that if $P \in K_4$, then P has no subposet isomorphic to either P_2 or P_5.

For the converse, suppose $P \notin K_4$, but any induced subposet of P is in K_4. Thus $P \neq B + C$, $P \neq B \oplus A_n$ and $P \neq A_n \oplus C$ for any posets B or C and any $n \geqslant 1$.

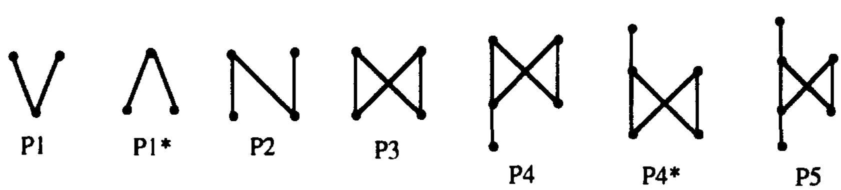

Fig. 3.

If P is not an SP poset, then P contains an induced subposet isomorphic to P_2 (by Theorem 7). Thus, we may suppose that P is an SP poset, so $P = Q \oplus R$ for SP posets Q and R.

Now P has some minimal element x and some non-minimal y such that the pair (x, y) is incomparable (or else $P = A_n \oplus S$ for some poset S and some $n \geqslant 1$). Let $z < y$ be minimal and note that x, y and z must all be distinct members of Q. Dually, there exist distinct u, v and $w \in P$ such that u and v are maximal, $u > w$ and the pair (v, w) is incomparable (or else $P = S \oplus A_n$ for some poset S and some $n \geqslant 1$) and u, v and $w \in R$. Then the six elements u, v, w, x, y and z form an induced subposet isomorphic to P_5, so we are done. □

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