Chaotic Attractors Near Forbidden Symmetry

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Abstract — We explore chaotic attractors with symmetries that are close to forbidden symmetries. In particular, attractors with planar crystallographic symmetry that contain an imperfect, but apparent, rotation that does not preserve the lattice of translations are constructed. This gives us a visual representation of patterns which have low symmetry type, but which contain additional, provocative structure. Like Penrose tilings and diffraction patterns of quasicrystals, our constructions include local rotations that are forbidden from being global by classical crystallographic theory.

1. INTRODUCTION

The fact that chaos is compatible with symmetry is intriguing and has been the subject of much recent study. In [7] families of functions which can be used to generate chaotic attractors with cyclic, dihedral and some of the planar crystallographic symmetries were explored. Chaotic attractors with symmetries from each of the frieze and planar crystallographic groups were determined in [2]. Various higher dimensional points groups have also been investigated [1,10,11]. Most relevant to the motivation for this investigation, previous work exploring the evolution of families of attractors that change from one symmetry type to another appears in [3]. It was observed there that visually interesting attractors with low symmetry type often occur when the attractor is in some sense near a higher symmetry type. Since the functions yielding many of the planar symmetry groups in [2] were obtained by masking the parameters of those from lower symmetry groups, it is natural to think of such a function as being close to having the higher symmetry if small values, like 0.05, were used instead of 0 in the mask. Thus, we take the view that both the actual symmetry and any nearby symmetries play a role in the qualitative appearance of attractors. In this note we use that same view as our motivation, but we will use a very different sense of nearness. We will investigate patterns that are near to having rotational symmetry.

It is a classical fact about planar crystallographic symmetry groups that only 2, 3, 4, and 6-fold rotations may appear [8,9]. Finding planar patterns and tilings with 5-fold rotations is a kind of Holy Grail. Work in that direction has included remarkable tilings, wave patterns and diffraction patterns of quasicrystals. Some striking tilings of Kepler's that involve pentagons and five-armed stars are reproduced in [8]. Complete planar aperiodic tilings may have local 5-fold rotational symmetry. Penrose tilings illustrate that structure [8,13]. The relation of forbidden symmetries to Cantor sets [6] and noncommutative geometry [4,5] have also been investigated. Wave patterns with these 5-fold symmetries have also been noted [14]. Diffraction patterns arising from quasicrystals have been found containing 10-fold and 12-fold rotational symmetry [13] and many other near regular patterns have been investigated [15]. Thus, patterns with rotational symmetry that is forbidden by the crystallographic symmetry groups are related to physical phenomena. Our techniques are not physically motivated, but they do produce patterns which have these “forbidden” rotational symmetries.

While [7] and [2] take somewhat different approaches toward creating chaotic attractors that include translations and rotations, they both generate illustrations that include 3-fold and 6-fold rotations by summing over 3 or 6 terms generated by the point group of rotations. Functions with 2-fold and 4-fold rotations may also be generated this way. Moreover, possible reflections that occur may be added by extending those sums to being over dihedral groups. In this note we investigate attractors which have rotations forced by sums of this type. In particular, we will see general methods for creating attractors with the symmetry of a lattice and other general techniques for generating functions with point group symmetry. When these are combined, the point group may or may not be compatible with the lattice. The next section deals with precisely describing these general functions. Nonetheless, when the rotations are incompatible with the lattice, interesting attractors may be constructed, some of which appear to be near forbidden symmetry. The last section explores attractors having these incompatible rotations.

2. FUNCTIONS WITH PLANAR CRYSTALLOGRAPHIC SYMMETRY

We begin by considering how to create functions that are candidates for having attractors with desired planar symmetry. A function $\mathrm{f}\colon \Re^2 \to \Re^2$ is said to be *equivariant* with respect to another function $\sigma\colon \Re^2 \to \Re^2$ if they commute: that is, $\sigma \circ \mathrm{f} = \mathrm{f} \circ \sigma$. We think of f as having the symmetry σ. If G is a group of transformations of the plane, then f is *equivariant* with respect to G if for all $\sigma \in G$, it is equivariant with respect to σ. It suffices to check equivariance on a set of generators of G.

We typically seek chaotic attractors with specified symmetry by doing Monte Carlo searches over parameters appearing in families of functions equivariant with respect to the required symmetries. First consider a lattice L in the plane along with a dual lattice L^*. This means that if $\vec{u} \in L$ and $\vec{v} \in L^*$, then $\vec{u} \cdot \vec{v}$ is an integer. We will often want integer multiples of 2π for our constructions and hence will use $\vec{v} \in 2\pi L^*$. That is, if $\vec{u} \in L$ and $\vec{v} \in 2\pi L^*$, then $\vec{u} \cdot \vec{v}$ is an integer multiple of 2π. Thus, $\cos(\vec{v} \cdot (\vec{x} + \vec{u})) = \cos(\vec{v} \cdot \vec{x})$ 1 and likewise for sine. This implies that if $\vec{\alpha}_{\vec{v}}, \vec{\beta}_{\vec{v}} \in \Re^2$ 2 are constant vectors, $\vec{\alpha}_{\vec{v}} \cos(\vec{v} \cdot \vec{x}) + \vec{\beta}_{\vec{v}} \sin(\vec{v} \cdot \vec{x}) \mod L$ is equivariant with respect to the translations defining L. Here the notation $\mathrm{mod}\ L$ is meant to designate reduction of points via the lattice to a fundamental cell. This gives a family of functions with the symmetry of a planar lattice which we summarize in the following proposition.

Proposition 1. If $\vec{\alpha}_{\vec{v}}, \vec{\beta}_{\vec{v}} \in \Re^2$ 3 are constant vectors and V is a finite subset of $2\pi L^*$ then

$$\mathrm{f}_V(\vec{x}) = \sum_{\vec{v} \in V} \left(\vec{\alpha}_{\vec{v}} \cos(\vec{v} \cdot \vec{x}) + \vec{\beta}_{\vec{v}} \sin(\vec{v} \cdot \vec{x}) \right) \mod L$$

is equivariant with respect to the translational symmetries of L.

Proof. We noted the fact for each term above and the property is preserved by the sum. ■

Notice that $\mathrm{f}_V(\vec{x})$ depends upon more than the notation would indicate; namely, it also depends on the lattice L, the fundamental cell for reduction $\mathrm{mod}\ L$, and the constants $\vec{\alpha}_{\vec{v}}$ and $\vec{\beta}_{\vec{v}}$.

In the next proposition we see how rotational symmetries may be obtained using sums over cyclic or dihedral groups.

Proposition 2. Let $f:\Re^2 \to \Re^2$ be an arbitrary function and let G be a finite group realized by 2 by 2 matrices acting on $\Re^2$ by multiplication on the right, then $h_f(\vec{x}) = \sum_{\sigma\in G} \sigma^{-1}(f(\sigma(\vec{x})))$ is equivariant with respect to G.

Proof. Let $\gamma \in G$. Then

$$h_f(\gamma(\vec{x})) = \sum_{\sigma\in G} \gamma(\sigma\gamma)^{-1}(f(\sigma(\gamma(\vec{x})))) = \gamma(\sum_{\sigma\in G} (\sigma\gamma)^{-1}(f((\sigma\gamma)(\vec{x})))) = \gamma(h_f(\vec{x}))$$

by the linearity of γ and the fact that $\sigma\gamma$ runs through G4 as σ does. ∎

Note that $h_f(\vec{x})$ also depends upon more than the notation would indicate; namely, it depends upon the representation of the group G. Functions of the form in Proposition 2 can be useful for creating attractors with point group symmetry. For example, the analog of this proposition in $\Re^3$ was used to create attractors with the symmetry of the dodecahedron [12]. However, we are interested in combining the functions in these propositions. Note that if we take a function of the form $h_{f_V}(\vec{x})$, where f_V has the form in Proposition 1, then by Proposition 2, $h_{f_V}(\vec{x})$ has the symmetry of G, but the symmetry of the lattice is likely to be gone. However, if the group is compatible with the lattice and we take the function $h_{f_V}(\vec{x}) \bmod L$, then all the symmetries are obtained.

Proposition 3. Let $f_V(\vec{x})$ be defined as in Proposition 1 and suppose G maps the lattice L back onto itself. Also let $h_{f_V}(\vec{x})$ be defined as in Proposition 2, then $h_{f_V}(\vec{x}) \bmod L$ is equivariant with respect to the translational symmetries of L and the symmetries of $G \bmod L$.

Proof. First, we check equivariance with respect to the lattice. Suppose $\vec{u} \in L$; we need to check that $h_{f_V}(\vec{x}+\vec{u}) \bmod L = h_{f_V}(\vec{x}) \bmod L$. Now by the definition of h_{f_V} and the linearity of elements of G,

$$\begin{aligned}
h_{f_V}(\vec{x}+\vec{u}) \bmod L &= \sum_{\sigma\in G} \sigma^{-1}(f_V(\sigma(\vec{x}+\vec{u}))) \bmod L \\
&= \sum_{\sigma\in G} \sigma^{-1}(f_V(\sigma(\vec{x})+\vec{u}')) \bmod L \text{ where } \vec{u}' = \sigma(\vec{u}) \in L \text{ since } G \text{ maps } L \text{ back onto itself} \\
&= \sum_{\sigma\in G} \sigma^{-1}(f_V(\sigma(\vec{x}))+\vec{u}') \bmod L \quad \text{by the equivariance of } f_V \text{ over } L \\
&= \sum_{\sigma\in G} \sigma^{-1}(f_V(\sigma(\vec{x}))) + \vec{u}'' \bmod L \quad \text{where } \vec{u}'' = \sigma^{-1}(\vec{u}') \in L \\
&= h_{f_V}(\vec{x}) \bmod L \quad \text{as required.}
\end{aligned}$$

Next, let $\gamma \in G$. We need to check that $h_{f_V}(\gamma(\vec{x}) \bmod L) \bmod L = \gamma(h_{f_V}(\vec{x}) \bmod L) \bmod L$. By Proposition 2 we have $h_{f_V}(\gamma(\vec{x})) = \gamma(h_{f_V}(\vec{x}))$ and hence $h_{f_V}(\gamma(\vec{x})) \bmod L = \gamma(h_{f_V}(\vec{x})) \bmod L$. The left hand side is $h_{f_V}(\gamma(\vec{x}) \bmod L) \bmod L$ by the first half of this proposition and hence we need only to check that

$\gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x})) \bmod L = \gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x}) \bmod L) \bmod L$. Now $\mathrm{h}_{\mathrm{f}_V}(\vec{x}) \bmod L = \mathrm{h}_{\mathrm{f}_V}(\vec{x}) + \vec{u}$ for some $\vec{u} \in L$. By the linearity of γ, we see that

$$\gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x}) \bmod L) = \gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x}) + \vec{u}) = \gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x})) + \gamma(\vec{u}).$$

Since G maps L back onto itself, $\gamma(\vec{u}) \in L$ and we see

$\gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x})) \bmod L = \gamma(\mathrm{h}_{\mathrm{f}_V}(\vec{x}) \bmod L) \bmod L$ as required. ■

For example, Figure 1 gives an illustration of this when $G = D_4$ and the lattice is generated by the standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$ which is self-dual. We used the finite set $V = 2\pi\{e_1, e_2, e_1 + e_2\}$. Notice the lattice symmetry, fourth turns and reflections are all apparent. In our Monte Carlo searches through parameter space, we only consider functions whose Ljapunov exponent is indicative of chaos. Thus, this figure illustrates a chaotic attractor with p4m planar crystallographic symmetry. Points visited a small percentage of the time are colored red. We increase the hue through magenta as the frequencies increase using a logarithmic bias on color change.

We note that these symmetries depend not just on G as an abstract group, but on the matrix representation of G. For example, if the lattice is the standard hexagonal lattice with generators e_1 and $e_3 = (-1/2, \sqrt{3}/2)$ and we use $G = D_3$ with generators $r = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then we get reflections across the x-axis and hence we get reflections through the corners of the standard fundamental cell but not through the interior third turns; this results in attractors with symmetry group p31m. On the other hand, if we use r and $t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to generate $G = D_3$, then the reflections do go through the interior third turns and hence we get an attractor with p3m1 symmetry. Illustrations of chaotic attractors with p3m1 and p31m symmetry created in this manner appear in [2]. Illustrations with p4m symmetry that were created in a different manner also appear there.

3. ATTRACTORS NEAR FORBIDDEN SYMMETRY

Now we turn to considering the case when the symmetry group is not compatible with the lattice. Our guarantees of equivariance via Proposition 3 are gone, but there is still the sense that these functions nearly have the symmetries of G since if the data happens to fall inside the fundamental cell, the symmetries of G are preserved. While the translational equivarance is also destroyed, the functions are sums of functions each of which would be equivariant with respect to translations of different lattices. That is, the terms in the sums corresponding to each fixed element of G would be equivariant with respect to the translations in some lattice, so that in practice attractors arising from such functions carry much coherence past the fundamental cell boundary.

Figure 2 shows an attractor discovered in one of our early experiments with using functions of the type $\mathrm{h}_{\mathrm{f}_V}$ when the group is incompatible with the lattice. We used the hexagonal lattice e_1, e_3 and took the fundamental cell to be $\{se_1 + te_3 | 0 \le s, t < 1\}$ along with the symmetry group $G = D_5$. The dual lattice is given by $e_4 = (1, -1/\sqrt{3})$ and $e_5 = (0, -2/\sqrt{3})$ and we used $V = 2\pi\{e_4, e_5, e_4 + e_5\}$. Notice that there are pairs of six-armed magenta hot spots. These are almost bilaterally symmetric and the pairs are related by fifth turns. This attractor seems to have lots of near mirrors and circular packing into the hexagonal lattice in addition to the partial fifth turn. Seeing similar images for $G = D_7$ where three hot spots set off by seventh turns were apparent suggested to us that centering our fundamental cell would be less destructive to the local symmetries. In particular, we switched to using $\{se_1 + te_3 | -1/2 \le s, t < 1/2\}$ as our fundamental cell for the hexagonal lattice.

Figure 3 and Figure 4 show illustrations of chaotic attractors created on the hexagonal lattice with $G = D_7$ and $G = D_5$ respectively. Notice that while the seventh turns in Figure 3 fade on the right, they are still quite apparent; they contain the expected mirrors and while there is almost a cell-like hexagonal packing, the interaction of neighboring regions can be observed. Figure 4 is our image that is nearest the pentagonal Holy Grail. Notice the clear lattice structure, fifth turns, and mirrors as expected. Various intertwining high frequency threads appear to tile the plane with various kinds of pentagons. While truly a p1 image, the image appears very close to having cm symmetry; in particular, note the near glide reflections. Also notice that the inner regions of the five sided stars do not keep clear 5-fold symmetry in the interior. It is the high frequency points that tend to best exhibit the probabilistic local symmetry.

Figure 5 shows an example on the square lattice using $G = C_5$. Of course the mirrors are missing but the fifth turns are clear and the intertwining of the cells remains interesting. It is also possible to use incompatible groups that yield attractors that have higher than p1 symmetry. For example, Figure 6 shows an illustration of an attractor that was given 8-fold turns using our techniques with $G = C_8$. This is compatible with the square lattice up to the symmetry group p4 which is the apparent symmetry of this chaotic attractor. The eighth turn is being locally forced and is not apparent, but neither is this a typical image of a chaotic attractor with p4 symmetry. The attractor seems to be intensely folded.

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REFERENCES

1. G. Brisson, K. Gartz, B. McCune, K. O'Brien, and C. Reiter. Symmetric attractors in three-dimensional space, *Chaos, Solitons & Fractals*, **7** 7, 1033-51 (1996).
2. N. Carter, R. Eagles, S. Grimes, A. Hahn, and C. Reiter. Chaotic attractors with discrete planar symmetries, *Chaos, Solitons & Fractals*, **9** 12, 2031-2054 (1998).
3. J. Dumont, F. Heiss, K. Jones, C. Reiter, and L. Vislocky. Chaotic attractors and evolving planar symmetry, *Computers & Graphics*, to appear.
4. M. S. El Naschie, Penrose Universe and Cantorian spacetime as a model for noncommutative quantum geometry, *Chaos, Solitons & Fractals*, **9** 6, 931-933 (1998).
5. M. S. El Naschie, Von Neumann Geometry and $\mathcal{E}^{(\infty)}$ quantum spacetime, *Chaos, Solitons & Fractals*, **9** 12, 2023-2030 (1998).
6. M. S. El Naschie, Forbidden symmetries, Cantor sets and hypothetical graphite, *Chaos, Solitons & Fractals*, **4** 12, 2269-2272 (1994).
7. M. Field and M. Golubitsky. *Symmetry in Chaos*. Oxford University Press, New York (1992).
8. B. Grünbaum, and G. Shephard. *Tilings and Patterns*. W. H. Freeman and Company, New York (1987).
9. T. Hahn, Editor. International Tables for Crystallography, Volume A. Kluwer Academic Publishers, Boston (1996).

10. C. Reiter. Attractors with the symmetry of the n-cube, Experimental Mathematics 5 4 327-336 (1996).
11. C. Reiter. Chaotic attractors with the symmetry of the tetrahedron, Computers & Graphics 21 6, 841-848 (1997).
12. C. Reiter. Chaotic attractors with the symmetry of the dodecahedron, submitted.
13. M. Senechal. *Crystalline Symmetries*. Adam Hilger, Bristol (1990).
14. M. Torres, G. Pastor, I. Jiménez, and F. Montero De Espinosa. Five-fold Quasicrystal-like Germinal Pattern in the Faraday Wave Experiment, *Chaos, Solitons & Fractals* **5** 11, 2089-2093 (1995).
15. G. M. Zaslavsky, R. Z. Sagdevv, D. A. Usikov, A.A. Chernikov. *Weak Chaos and Quasi-regular patterns*. Cambridge University Press, Cambridge (1991).

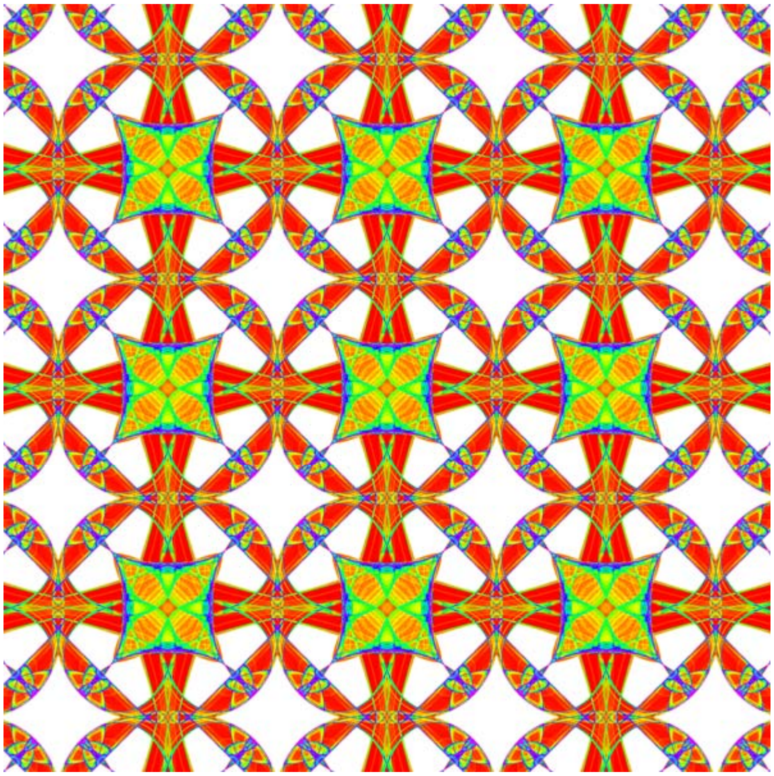

Figure 1. A chaotic attractor with p4m symmetry.

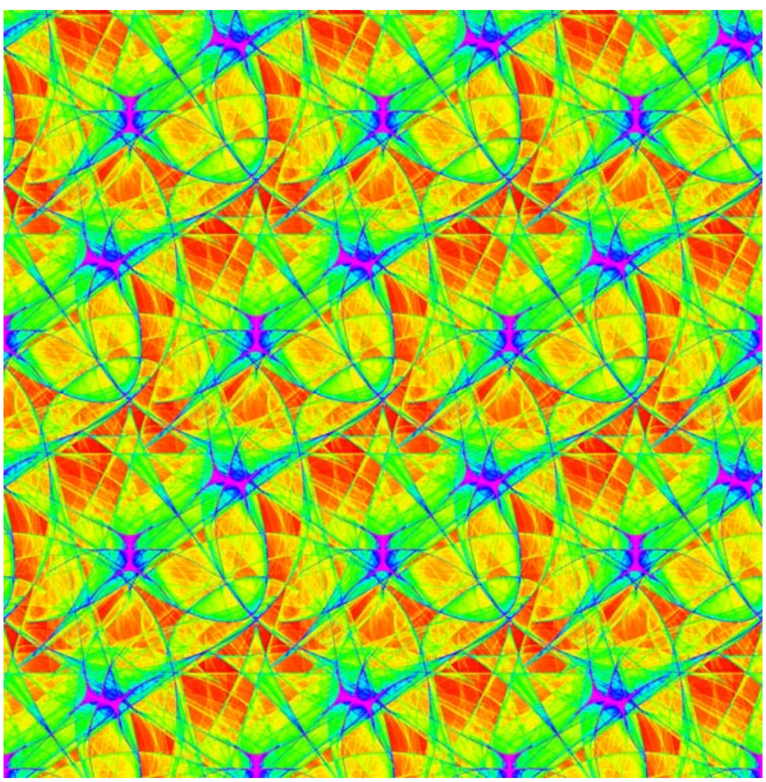

Figure 2. A chaotic attractor on a hexagonal lattice with uncentered 5-fold rotations and mirrors.

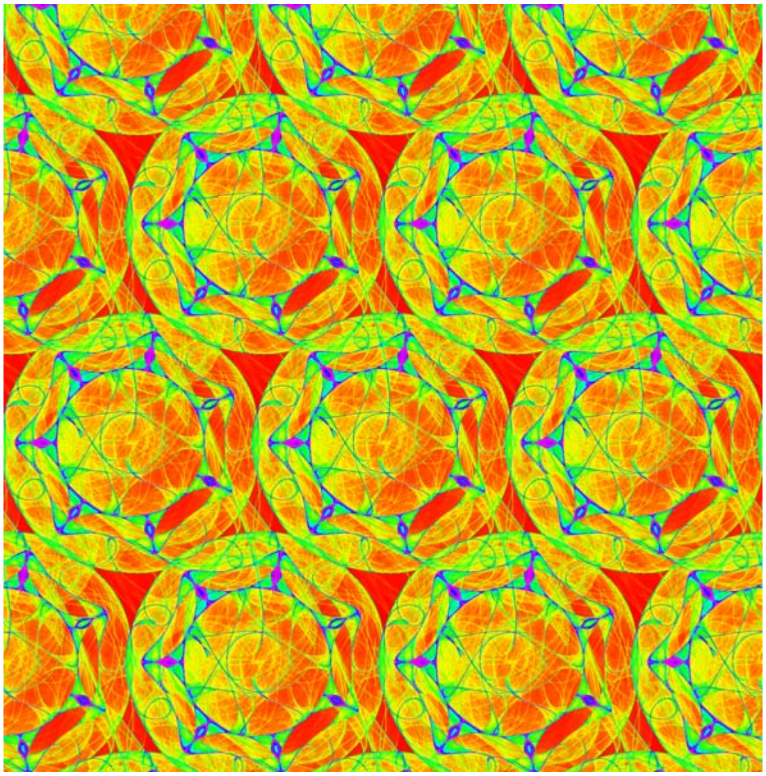
Figure 3. A chaotic attractor on a hexagonal lattice with centered 7-fold rotations and mirrors.

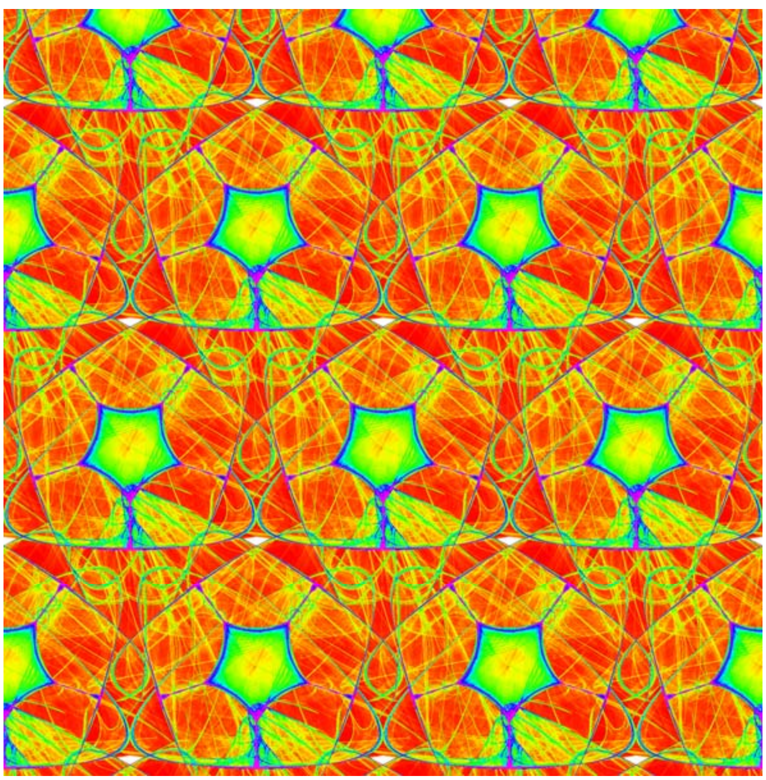
Figure 4. A chaotic attractor on a hexagonal lattice with centered 5-fold rotations and mirrors.

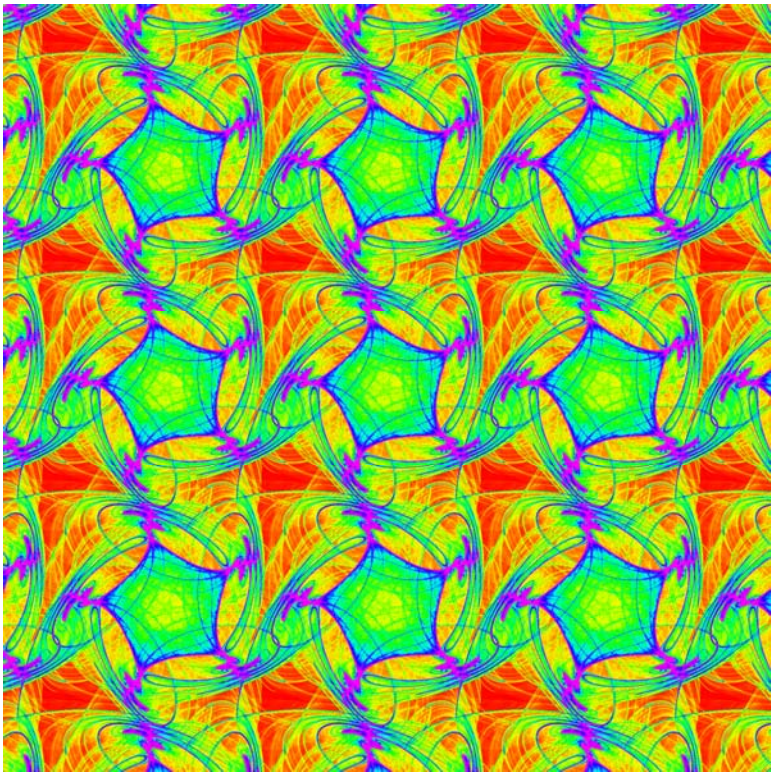

Figure 5. A chaotic attractor on a square lattice with centered 5-fold rotations.

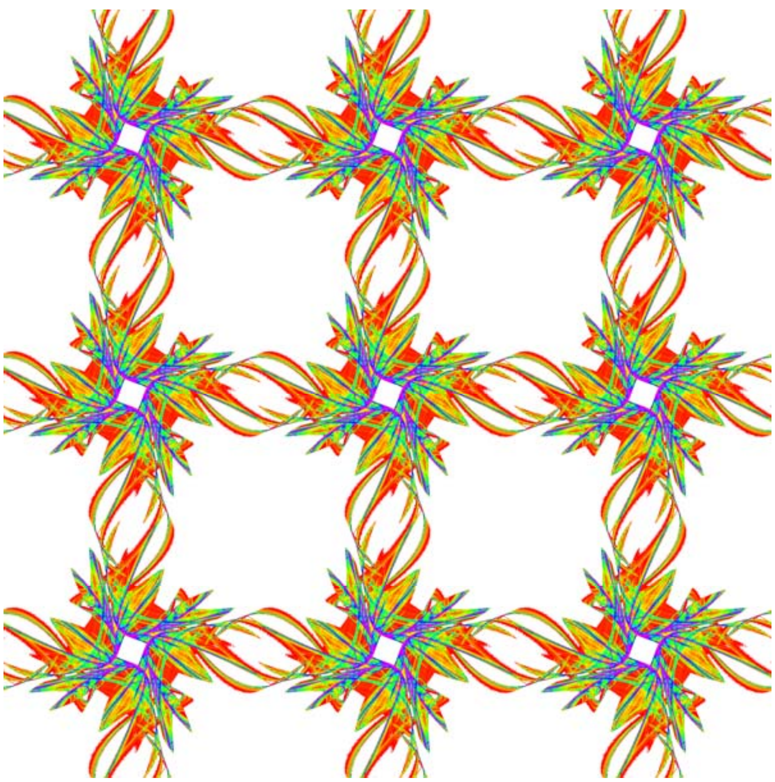

Figure 6. A chaotic attractor on a square lattice with p4 symmetry created using centered 8-fold rotations.