Chaotic Attractors with the Symmetry of the Dodecahedron

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Polynomial functions in three space that are equivariant with respect to the symmetries of a dodecahedron are determined. These are constructed using a sum of functions under changes of coordinates with respect to the group. Monte Carlo searches through parameter space lead to functions that are experimentally chaotic and which have visually appealing attractors. Ray tracing techniques allow for the three dimensional attractors to be rendered so that both their complexity and symmetry may be observed.

Keywords: Attractor – Chaos – Dodecahedron – Icosahedron – Ray Tracing

1. Introduction

Chaotic attractors have arisen in the study of physical dynamics and remain fascinating because of their complex behavior. One intriguing aspect of chaotic attractors is that they can have much symmetry without losing their chaotic nature. Recent work includes [4] which explains how to construct attractors with dihedral and cyclic symmetry and some of the planar wallpaper symmetries. Chaotic attractors with the symmetry of each of the discrete planar symmetry groups are illustrated in [2]. Attractors with the symmetry of point groups in higher dimensional space have included those with the symmetry of the cube [1], hypercube [7] and tetrahedron [8]. This short note describes the construction of chaotic attractors with the remaining 3-dimensional point group symmetry. Namely, with the symmetry group of the dodecahedron and icosahedron. This group has remained elusive because of the complexity of restricting maps to respect the symmetry of fifth turns in three space. Moreover, unlike those previous illustrations in 3 space, the images here are rendered using ray tracing and an animation is created.

The technique described in this short note utilizes sums of polynomials under the transformations of coordinates arising from the symmetry group in 3-dimensions. This bypasses the difficulty in finding the restricting maps. While a variant of this idea is implicit in techniques used for finding chaotic attractors with the symmetry of various hexagonal lattices, this application to a point group appears to be unlike earlier experimental explorations of chaotic attractors with the symmetry of a point group. Originally we attempted to use restriction techniques like those used for other point groups in space as in [8]. However, for that approach it was natural to embed the problem in at least dimension 5 in order to realize the symmetry group of the dodecahedron using only permutations of coordinates and sign changes. While direct determination of the polynomials with the required symmetry in this way is possible, the constructions are complex and projecting attractors from higher dimensions to 3-dimensions in a manner that visually preserves the symmetry remains a serious difficulty. The author had greatest preliminary success in dimension 6, where involution through the origin was natural, and this led to somewhat fruitful experiments. However, the approach we will follow in this note has been much more successful. Moreover, the success of this technique on a point group in space provokes the question, currently under investigation, of whether similar techniques would lead to qualitatively more interesting illustrations of attractors for other point groups.

2. Functions with the symmetry of a dodecahedron

The symmetry group of the dodecahedron is the same as that of the icosahedron and is often denoted by I. This rotation group has 60 elements [5] and one can check that if the vertices of the icosahedron are chosen in the style of [3] to be $(0,\pm\tau,\pm 1)$ 1, $(\pm 1,0,\pm\tau)$ 2 and $(\pm\tau,\pm 1,0)$ 3 where $\tau = (1+\sqrt{5})/2$ is the golden ratio4, then the transformations $\mathrm{T}_1(\vec{x}) = \frac{1}{2}\begin{pmatrix} 1 & \tau & \tau^{-1} \\ -\tau & \tau^{-1} & 1 \\ \tau^{-1} & -1 & \tau \end{pmatrix}(\vec{x})$ 5 and $\mathrm{T}_2(\vec{x}) = \frac{1}{2}\begin{pmatrix} \tau & \tau^{-1} & -1 \\ \tau^{-1} & 1 & \tau \\ 1 & -\tau & \tau^{-1} \end{pmatrix}(\vec{x})$ 6 generate I. If one desires the symmetry to also include reflections,

Comment [m1]:

then one can add $T_3(\vec{x}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(\vec{x})$ as a generator as well. This full symmetry group that includes reflections has 120 elements and we designate it by $\bar{\mathrm{I}}$.

In order to construct chaotic attractors with specified symmetry, we seek classes of simple, but nonlinear functions that are equivariant with respect to the desired symmetry. That is, we want nontrivial functions $f: \Re^3 \to \Re^3$ 7 satisfying $f \circ T_1 = T_1 \circ f$ 8 and $f \circ T_2 = T_2 \circ f$ and perhaps $f \circ T_3 = T_3 \circ f$ in order to generate the symmetries in I or $\bar{\mathrm{I}}$ 9. This can be accomplished by taking functions of the form described in the following lemma:

Lemma 1: Let $P: \Re^n \to \Re^n$ be an arbitrary function and let *G* be a finite group realized by *n* by *n* matrices acting on $\Re^n$ by multiplication on the right, then $f(\vec{x}) = \sum_{\sigma \in G} \sigma^{-1}(P(\sigma(\vec{x})))$ is equivariant with respect to *G*.

Proof: Let $\gamma \in G$. Then

$$f(\gamma(\vec{x})) = \sum_{\sigma \in G} \gamma(\sigma\gamma)^{-1}(P(\sigma(\gamma(\vec{x})))) = \gamma(\sum_{\sigma \in G} (\sigma\gamma)^{-1}(P((\sigma\gamma)(\vec{x})))) = \gamma(f(\vec{x}))$$

by the linearity of γ and the fact that $\sigma\gamma$ runs through *G*10 as σ does.

For the illustrations in this short note, we constructed functions of the type in Lemma 1 for $G = \bar{\mathrm{I}}$.

3. Illustrations

We took the function P described in Lemma 1 to be a three coordinate polynomial with *x*, *y* and *z* raised to the powers 0, 1 and 2 in all possible products. This leads to 81 coefficients a_{ijkm} where *i* corresponds to the power of *z*, *j* to the power of *y*, *k* to the power of *x* and *m* to the output coordinate. In our Monte Carlo search we selected these coefficients randomly between -0.15 an 0.15. These parameters were chosen after some experimentation in order to frequently give an attractor with Ljapunov exponent between 0.01 and 0.6. A positive Ljapunov exponent is indicative of chaos and some literature has suggested Ljapunov exponents in the range 0.01 to 0.6 are aesthetically pleasing [9]. We also required our functions to satisfy simple tests for nonperiodicity (iterate 199 does not appear in iterates 0-198) and noncollinearity (the iterates 190-199 don't fall on a line). We attempted utilizing only the powers 0 and 1 which leads to simpler polynomials that are nonlinear in one mixed term. We did not have success in locating many interesting chaotic attractors with that lower degree function.

Figure 1 shows an attractor with the symmetry of $\bar{I}$. Our code was implemented in J [6] and the parameters for the function used to create Figure 1 are given in Appendix 1. A few lines of J code that would allow readers to duplicate the function are also included there. We resolved space surrounding the attractor into an 800 by 800 by 800 grid and kept sorted lists of positions visited and frequencies. The points were then sorted by frequency and piped from J into an isosurface "blob" POVRAY object that adds some coherence not obtained from using spheres or boxes. POVRAY [10] is available from www.povray.org. Color in the image corresponds to the frequency with which the position was visited. Colors run through the hues from red to magenta. In particular, red in the image corresponds to regions visited only a few times and yellow, green, cyan, blue and magenta correspond to increasing frequencies of visitation. Thus the magenta threads which have the dodecahedral shape are the regions visited most. This attractor has low probability regimes which are cloudlike and dispersed in a manner that obscures the interior. Thus we have used a threshold of ignoring points visited 5 or fewer times out of the approximately 50 million points that we resolved. We ordinarily use a logarithmic bias in determining how quickly to change color [8], but we have used a linear scale for this figure in order to accentuate the contrast and the dodecahedral shape. An animation of this attractor may be found at www.lafayette.edu/~reiterc/dodec/index.html.

Figure 2 shows an attractor with the same symmetry group, but there are twelve "hot spots" corresponding to the vertices of an icosahedron. Note that the attractors have threads appearing on and inside twelve ballish regions as well as more complex mixing in the interior. Notice that ten threads appear to cut through each hot spot. We used our traditional logarithmic bias in color changes for this image and again used a frequency threshold, although in this case it was not essential.

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Appendix 1

The following lines of J code provide a complete implementation of the function used to create the attractor in Figure 1. Appendix 2 gives a pseudo-code/mathematical description of this same function. The last entries in this appendix give several iterations of the function that can be used as a check against alternate implementations. Users can duplicate these experiments using a copy of J from www.jsoftware.com. The J code in this appendix is can be found at www.lafayette.edu/~reiterc/dodec/index.html. First we create the matrix group I with its 120 entries and the corresponding inverses II using group product, prods, repeatedly on the generators T1, T2 and T3.

```
x=:+/ . *
prods=:\:~@~.@(* >&1e_14@:|)@(,(,/)@:( x"2/~))
ri=:%r=.-:>:%:5
```

```
   T1=:-:3 3 $1,r,ri,(-r),ri,1,ri,_1,r
   T2=:-:3 3$r,ri,_1,ri,1,r,1,(-r),ri
   T3=:_1 1 1*=i.3

   $I=:prods^:3 T1,T2,:T3
120 3 3
   $II=:%. I
120 3 3
```

Next we create the dodecahedral function builder DD and input the 81 parameters. Lastly, we create the function G used to create Figure 1 and exhibit iterations 0 to 4.

```
   DD=:1 : '+/@(II"_ x"2 1 ])@((0&{ x 1&{ x 2&{ x
                               m."_)@(^/&(i.#m.))"1@(I&x)) f.'

   pars=:3 3 3 3$,".;._2]0 : 0
   0.0180157305   0.0903833937  _0.1262988060
   0.0959720238  _0.0981938805  _0.0445451871
   0.1290451044  _0.1389270186  _0.0463975410
  _0.1034676072   0.1199258382  _0.1064360991
   0.0284841300  _0.0672225822  _0.0099387726
   0.0590534490   0.1113198750  _0.1468569510
  _0.1247709900  _0.0260266215  _0.0294227928
  _0.1088782995   0.0824206119   0.1432259172
  _0.0020062041  _0.1182703734   0.0298362312
  _0.1424609703  _0.0415270104  _0.1444622841
  _0.0776052450   0.0886481373   0.1092472107
   0.1178740806  _0.0903261324  _0.1113043062
   0.1085292588   0.0512562201  _0.1367073846
   0.0589886136  _0.0783689961  _0.1477134882
  _0.1205961711  _0.0598447191   0.0898105986
   0.1467316815   0.1193712204  _0.1278978093
  _0.0784800531   0.0857490972  _0.0149189145
   0.0578068920  _0.1395622734   0.0768726459
  _0.1014373995   0.0416295027   0.0670539840
  _0.1236866562  _0.1016276886   0.1434392721
  _0.0161524251   0.0261961311  _0.1216222695
   0.0945181746   0.0669648237  _0.1222031436
  _0.0682310184   0.1412775246  _0.0486418341
  _0.0233046243   0.1191796536  _0.0475573362
  _0.0961468965  _0.1408879608  _0.0039553878
   0.1218001257  _0.1052849598  _0.1243178283
   0.0902625387  _0.0575116383   0.0018951813
```

```
)

   G=:pars DD
   G^:(i.5) 0.1 0.2 0.3
0.1      0.2      0.3
0.253974 0.507183 0.761566
0.30382  0.527163 0.874296
0.234707 0.414627 0.717742
0.313538 0.570422 0.994186
```

Appendix 2

In this appendix we describe the function constructed in Appendix 1 using traditional notation. We begin by letting I_1, I_2, and I_3 be the three by three matrices used to define the three transformations T_1, T_2, and T_3 in Section 2. We use matrix multiplication to compose these three matrices and repeat that on the resulting matrices until the set stabilizes. The result is the dodecahedral symmetry group represented by 120 three by three matrices which we will denote I_i for $1 \le i \le 120$. Now let a_{ijkm} denote the 3 by 3 by 3 by 3 four-dimensional array of parameters given in Appendix 1. We can define a polynomial map from $P:\Re^3 \to \Re^3$ by $P(x,y,z)_m = \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{2} a_{ijkm} x^k y^j z^i$ where the subscript $0 \le m \le 2$ runs through the output coordinates in $\Re^3$. Then we create our function with the desired symmetry by $G(\vec{x}) = \sum_{n=1}^{120} I_n^{-1} \cdot P(I_n \cdot \vec{x})$ where the dots indicate a matrix times vector product.

Biographical Sketch:

Cliff Reiter is an Associate Professor of Mathematics at Lafayette College. He is enthusiastic about using high level programming languages for teaching and research in mathematics. His research areas include elementary number theory and topics in dynamics and mathematical visualization. He is the author of the text *Fractals, Visualization and J*. When not doing mathematics he is likely to be hiking with family or friends in the high peaks region of the Adirondacks.

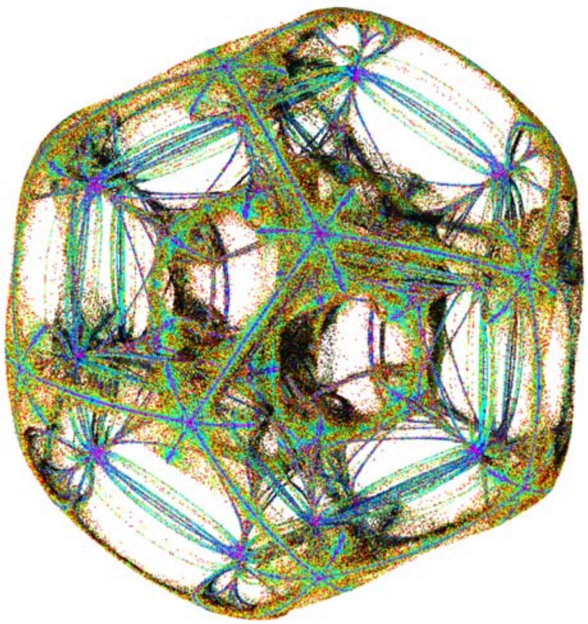

Figure 1. A chaotic attractor with the symmetry of a dodecahedron.

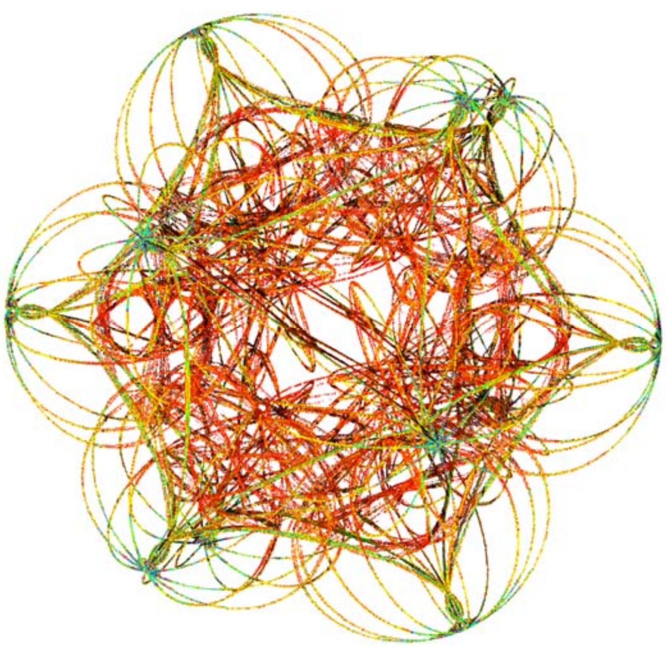

Figure 2. A chaotic attractor with the symmetry of a dodecahedron and the form of an icosahedron.