Fuzzy Automata and Life

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1. Introduction

Cellular Automata have generated much interest [1,2,3] because of their diverse behavior and usefulness as a discrete model for many processes. Wolfram's 1984 paper on universality and complexity in cellular automata [3,4] described four classes of behavior for automata: class 1 for homogeneous stable behavior, class 2 for simple periodic patterns, class 3 for chaotic aperiodic behavior and class 4 for complex behavior which generates local structures. Recent work by Cattaneo, Flocchini, Mauri, Quaranta Vogliotti, and Santoro [5] introduced an alternate classification scheme that can be obtained by generalizing Boolean cellular automata to a fuzzy automata and then observing the qualitative behavior of a Boolean window embedded in a fuzzy domain. These automata can be classified according to whether the Boolean behavior dominates, a mixture of Boolean and Fuzzy behavior appears, or purely fuzzy behavior appears. The fuzzy behavior can further be divided into subclasses with homogeneous and nonhomogeneous behavior.

In this paper we further investigate these fuzzy extensions of one-dimensional automata and we also consider two-dimensional automata. We investigate the impact of using several choices for the fuzzy logic functions including the unusual logic used in [5]. The choice of different fuzzy logics lead to significantly different behaviors. We also introduce and use the one-dimensional continuous dynamics of the response functions to explain much of the qualitative behavior of these fuzzy automata. Lastly, we consider two-dimensional automata giving the Game of Life. We will see the greatest common divisor/least common multiple trivalent logic seems to be remarkably able to nurture the generation of Life without dominating it; the Game of Life appears even more complex in this context. The situation is somewhat analogous to a wave tank with expanding walls generating waves with the interference controlled by the rules of the Game of Life.

The authors of [5] defined a quantity they call rule entropy in a simple manner based upon the defining Boolean rule. While of limited capacity compared to more classical measures of entropy [3,6], the simplicity of this entropy comes from the fact that it does not depend upon the configuration of states. They observed that this entropy was bounded between 0 and 6 for their automata and that class 3 (chaotic) behavior appeared to be limited to their fuzzy classes and associated with intermediate or high values of rule entropy, thus connecting the qualitative behaviors with this simple quantitative measure. In the Appendix we examine rule entropy noting corrections to the tables of [5] and examining a natural extension to the Game of Life.

This paper shows that the generalization of ordinary automata to fuzzy automata can be accomplished using a wide variety of fuzzy logical systems. The qualitative behaviors are remarkably different although the dynamics of response functions offers considerable insight into the long term behavior. While the fuzzy automata discussed here are intriguing in their own right, it is hoped that this work provokes further interest in investigating how to best use the continuous dynamics available from these fuzzy automata and exploring which fuzzy logics best capture the qualitative behavior of their Boolean forerunners.

2. Fuzzy Automata

Classical cellular automata consist of an array of states, typically selected from a finite set, along with local rules for updating the array of states. Some of the simplest automata are defined on a one-dimensional lattice of cells, are Boolean (two state), and base the future state of a cell upon the states in a 3-cell neighborhood consisting of the cell and its left and right neighbors. Let a_i^t denote the value, 0 or 1, of the cell at position *i* at time *t*. Table I shows the behavior of an example automaton. It is called Rule 17 since the list of results, a_i^{t+1}, gives the binary digits of 17.

Rule 17 can be readily transformed into disjunctive normal form by selecting a variable or its negation for each of the three input states for each possible 1 output. Thus, we get

$$a_i^{t+1} = (\overline{a}_{i-1}^t \wedge \overline{a}_i^t \wedge \overline{a}_{i+1}^t) \vee (a_{i-1}^t \wedge \overline{a}_i^t \wedge \overline{a}_{i+1}^t)$$

where we denote logical "not" with an overbar. Note that we can capture the essential details of this representation with the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The rows of the matrix are the two input rows in Table I that have output value 1.

Now we turn this into a fuzzy rule by reinterpreting the logical functions in that formula. In all of our versions of fuzzy automata we will imagine 0 as false (or dead), 1 as true (or alive) and hence logical "not *x* " is consistent with 1-*x*. In [5], logical "*x* or *y* " was replaced by min{1, *x*+*y*} while "*x* and *y* " is given by multiplication: *x* * *y*. This allows us to view the disjunctive normal form for Rule 17 as a formula where cells have values from the interval [0,1] in a way that is consistent with the Boolean interpretation on the endpoints 0 and 1. Hence, we have a fuzzy automaton. We discuss several other fuzzy automata in Section 4 and [7] offer a discussion and visual presentation of some other fuzzy logics.

a_{i-1}^t	a_i^t	a_{i+1}^t	a_i^{t+1}
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Table I. The definition of Rule 17.

The time evolution of these fuzzy automata can be visualized graphically. In Figure 1 we see our first four examples of automata of these type. In particular, we have illustrations of the behavior of Rules 17, 125, 73, and 146 on a random Boolean window embedded in an infinite fuzzy background with value one-half. The Boolean window is 64 units wide and the specific random choices used can be duplicated with the J [8] command `?.64$2`. Implementation of the automata appears in [9] and additional rules appear on [10]. The initial values are the first row in the figure and the states of subsequent stages of the automaton appear in subsequent rows. The palette scheme used for our automata is shown in Figure 2. White is used for value 0 (dead cells), black is used for value 1 (alive cells) and intermediate values are shown in hues that alternate, for contrast, but generally move from lighter hues (cyan, magenta, yellow) to darker hues (red, green, blue) as the fuzzy value gets nearer to one.

3. Repeated Input Response Function

If we let $\mathrm{f} : \Re^3 \to \Re$ denote the local rule on a fuzzy one dimensional automata, we can define a diagonal map that we will call the repeated input response function, or simply the response function, by $\mathrm{r}(x) = \mathrm{f}(x, x, x)$. Since $\mathrm{r} : \Re \to \Re$ we can investigate the real dynamics of this response function. The response function is undoubtedly too simple to capture all the dynamics of $\mathrm{f}(x, y, z)$; nonetheless, we will see it is remarkably useful for predicting the long term behavior of our fuzzy automata.

Figure 3 shows the response map for several fuzzy automaton rules using the fuzzy logic described in the previous section. Notice the functions show great variety. The intersection with the line $y = x$ gives the fixed points. We see Rule 17 has a repelling fixed point near 0.4 while 0 and 1 form an attractive two cycle. The oscillations near 0 and 1 appearing in Figure 1 for Rule 17 are expected. The response function for Rule 125 has a very slowly attracting fixed point near 0.6, hence the damped oscillations near 0.6 observed in Figure 1. The response function for Rule 73 has a strongly attractive fixed point near 0.4 and hence the homogeneous fuzzy region stabilizes quickly. The response function for Rule 146 has three fixed points, 0, 1/3 and 1. While the uniform fuzzy background quickly converges to 1/3, the complexity of the fuzzy rule is reasonable.

In Table II we see the qualitative behavior of these fuzzy automata are enumerated with a summary of the dynamics of the response function and a list of the automata with those given characteristics. The classes are organized outermost by whether the background converges in the long term to homogeneous value equal to 0, an oscillation between 0 and 1, a homogeneous value between 0 and 1 or homogeneous value equal to 1. These situations are further divided into whether there is a Boolean window (BW), whether the window travels (trav), whether there is an expanding window (exp), or whether there is an expanding fuzzy window on one side (exp fz1) or two sides (exp fz). The behavior at 0 and 1 is described by the values of the response function at those points with a plus sign next to attractive fixed points. Thus 01+ denotes fixed points at 0 and 1 with 1 being attractive. Also +(10) denotes an attractive 2-cycle on those values and +F and -F denote attracting and repelling intermediate (fuzzy) fixed points. The categories "all" and "all2" occur when all fuzzy values are fixed points or in a 2-cycle, respectively.

Notice that all the fuzzy automata with -F match those with 2-cycle long term behavior except for Rule 23 which has an expanding two cycle window in a fuzzy background and Rule 232 which has an expanding Boolean window in a fuzzy background—which make those barely exceptional. The automata with no fuzzy fixed points are exactly those with long term homogeneous 0 or 1 behavior. The automata with fuzzy long term behavior are those whose response functions have attractive fuzzy fixed points, except the two cases 23 & 232 mentioned earlier. Distinguishing the behaviors of the subcategories of fuzzy behavior is more difficult, but in general having "all" or "all2" behavior in the response function is associated with rich behavior. Even in the case when the homogeneous fuzzy behavior completely dominates, the slope of the response function has predictive value. Namely, when the response function has a slope near ±1 at an attractive fixed point, then the nonhomogeneous fuzzy behavior tends to persist longer, as one might expect. These slopes are not apparent from the table.

4. Other Fuzzy Logics

We consider several other fuzzy logics in order to discuss their use in fuzzy automata. See [11] for a discussion of some classical multi-valued logics. Table III summarizes the five logics that we will discuss. These logics were chosen because they appear in other literature [5,11,12,13] and they seem to have interesting fuzzy dynamics. In previous sections, our illustrations have so far considered only the fuzzy logic used in [5], Logic 1 in our table.

Homogeneous 0	+00		0, 8, 32, 40, 56, 64, 74, 88, 96, 98,
	+01		128, 136, 160, 168, 192, 224,
BW	+00		4x, 12x, 36, 44, 68x, 72, 76, 100, 104,
	+01		132x, 140x, 164, 196x, 200,
trav	+00		2x, 10x, 16x, 24x, 34x, 42x, 48x, 66x, 80x, 112x, 130,
	+01		138x, 144x, 152, 162x, 176x, 194x, 208x,
2-Cy 0-1	+(10)	-F	1, 3, 5x, 7x, 17, 19, 21x, 31x, 55x, 63, 87, 95x, 119, 127,
Fuzzy	00	+F	6, 20, 22, 26, 30, 38, 52, 54, 60, 62, 82, 86, 90, 102, 106, 110, 118, 120, 122, 124, 126,
	01	+F	148, 150, 154, 158, 166, 180, 210, 214,
	10	+F	9, 25, 33, 35, 37, 41, 45, 49, 57, 59, 61, 65, 67, 75, 89, 91, 97, 99, 101, 103, 105, 107, 111, 115, 121, 123, 125, 134,
	11	+F	129, 131, 135, 137, 145, 147, 149, 151, 153, 155, 159, 161, 165, 167, 169, 181, 193, 195, 211, 215, 225,
exp fz	00	+F	14, 18, 46, 58, 84, 114, 116,
	01	+F	142, 146, 182, 212,
	01	all	184, 226,
	10	+F	11, 43, 47, 81, 113, 117,
	10	all2	27, 39, 53, 83,
	11	+F	139, 143, 163, 177, 183, 209, 213,
BW	00	+F	28x, 70x, 108,
	01	+F	156, 198,
	01	all	204,
	10	+F	73x, 109x,
	10	all2	51,
	11	+F	157x, 199x, 201,
exp	00	+F	50,
	01	+F	178,
	+(10)	-F	23,
	+01+	-F	232,
	10	+F	13, 69, 77, 79, 93,
	11	+F	179,
trav	01	all	170, 240,
	10	all2	15, 85,
expfz1	00	+F	78, 92,
	01	all	172, 202, 216, 228,
	11	+F	141, 197,
exp fz	00	+F	94,
	10	all2	29, 71,
	11	+F	133,
Homogeneous 1	01+		234, 238, 248, 250, 252, 254,
	11+		173, 185, 227, 229, 235, 239, 249, 251, 253, 255,
BW	01+		206, 218, 220, 222, 236,
	11+		203, 205, 207, 217, 219, 221, 223, 233, 237,
trav	01+		174, 186x, 188, 190, 230, 242, 244, 246,
	11+		171, 175, 187x, 189, 191, 231, 241, 243, 245, 247

Table II. Long term behavior of Logic 1 fuzzy automata

		x or *y*	*x* and *y*
Logic 1	CFMQVS	min(1, *x*+*y*)	*x* * *y*
Logic 2	Max/Min	max(*x*, *y*)	min(*x*, *y*)
Logic 3	Probabilistic	*x* + *y* - *x* * *y*	*x* * *y*
Logic 4	MV	min(1, *x*+*y*)	max(0, *x* + *y* -1)
Logic 5	gcd/lcm	gcd(*x*, *y*)	lcm(*x*, *y*)

Table III. Five Fuzzy Logics

Notice that Logic 1 uses a combination of logical functions from Logics 3 and 4. Table IV gives some values of these fuzzy logics. Logic 1 has an "or" that is fairly high valued while the "and" is fairly low valued. For all the examples in this paper the only initial values are 0, 1/2, and 1. Notice that when Logic 1 is restricted to those three values, it is not closed and hence doesn't form a trivalent logic.

The most classical multivalued logic is Logic 2, which is based on maximum replacing "or" and minimum replacing "and". When restricted to the three initial values, the classical trivalent Lukasiewicz Logic results [12]. Logic 3 is a probabilistic Logic which does not restrict to a trivalent logic. It shares "and" with Logic 1, but "or" has values that are sometimes lower. Logic 4 restricts to a trivalent logic, but not any of those mentioned in [12]. The "or" values are shared with Logic 1 but there are many more 0 values in the evaluation of "and".

Logic 5 is not a logic on the interval [0,1] since it does not map back to the interval [0,1]. However, its trivalent restriction is closed which makes it a trivalent logic when restricted to our initial values. Its logical "or" is the same as in the Bochvar trivalent logic while the "and" is not listed in [12]. We were motivated to consider these gcd/lcm functions as extensions of or/and since the gcd/lcm extension of or/and is a compatible extension and many of the same identities are satisfied by these function pairs. We have also previously found some remarkable connections between classical Sierpinski gaskets, carpets and higher dimensional analogues when one extends or/and to gcd/lcm [13].

The qualitative behavior of the versions of fuzzy automata that are produced by these logics are diverse. We saw the behavior of rules 17, 125, 73, and 146 with Logic 1 in Figure 1. We believe the diversity of this fuzzy automata results in part from two facts. First the logic does not restrict to a trivalent logic so that values tend to mix. Second, the repeated input response function, as described, has rich behavior for many of these automata.

While Logic 2 represents a very common multi-valued logic, one-dimensional fuzzy automata produced with this rule tend to have relatively simple behavior. This is forced by the fact that the rule restricts to a trivalent rule. Nonetheless, automata can be classified according to whether fuzzy behavior dominates or a Boolean region persists. The Boolean regions may have vertical or diagonal boundaries. As for Figure 1, the initial configuration is a random Boolean window in a fuzzy background of level 0.5. The palette is again given in Figure 2.

Figure 4 shows the behavior of these automata using Logic 2 with Rules 17, 125, 73, and 146. Figure 5 shows Logic 3 used for these rules. The behavior of these probabilistic automata is often quite close to the behavior seen for Logic 1 but there is a greater tendency toward homogeneous fuzzy behavior. Logic 1 is preferable to Logic 3 if greater qualitative diversity is sought. On the other hand, Logic 3 has more uniform definitions and would likely be more amenable to analytic study. Even so, in some cases, Logic 3 behavior is quite interesting. For example, Rule 50 leads to an undramatic expanding Boolean window using Logic 1, and hence represents relatively simple behavior. However, Figure 6 shows Rule 50 for Logic 3 where very rich behavior appears.

Logic 4 is trivalent when restricted to our initial values and it generates some interesting behaviors as illustrated in Figure 7. Indeed, these behaviors are the most interesting trivalent extension that we have seen. Logic 4 on 0 and 0.5 seems closely related to the corresponding Boolean rule. In particular, Rule 146 restricted to 0 and 0.5 behaves the same as Rule 146 on 0 1 except for the case when all three cells are 1. So actually, Rule 146 restricted to 0 and 0.5 behaves the same as Rule 18 on Boolean values and the Sierpinski triangle features are known to arise from that automaton. Given these remarks, it is perhaps not surprising that the mixture of Logic 3 and 4 used in Logic 1 seems to create the most diversity. That in turn makes it a good candidate for classification of quantitative behavior.

Lastly, Logic 5 is also trivalent on our initial values and has relatively modest behavior when used for one dimensional automata. The fuzzy values do not dominate and they play little role except to provide a window in which the Boolean values can grow. Figure 8 shows Boolean rules periodically extended and initially exhibiting three periods. Contrast those purely Boolean rules with the Logic 5 fuzzy rules, shown in Figure 9, exhibiting the evolution of a Boolean window embedded in an infinite 0.5 domain, as we have studied in our previous examples of fuzzy automata. In Figure 9, the expanding Boolean regions have more of the global feel of the automata, and seem richer than the purely Boolean examples. Indeed we will see below that Logic 5 provides a remarkable growing environment for the Game of Life.

5. The Game of Life

The Game of Life [14,15,16] is a two dimensional Boolean automaton that has local rule defined on three by three neighborhoods surrounding a cell. A cell is alive at the next generation if it is currently alive and has two or three of its eight neighbors alive or if it is currently dead and has exactly three of its eight neighbors alive. Now we can list the eight neighbors of $a_{i\,j}^t$ and write the rule for Life in disjunctive normal form which contains terms like those that follow.

$$
\begin{aligned}
a_{i\,j}^{t+1} = {} & (a_{i-1\,j-1}^t \wedge a_{i-1\,j}^t \wedge a_{i-1\,j+1}^t \wedge \bar{a}_{i\,j-1}^t \wedge \bar{a}_{i\,j}^t \wedge \bar{a}_{i\,j+1}^t \wedge \bar{a}_{i+1\,j-1}^t \wedge \bar{a}_{i+1\,j}^t \wedge \bar{a}_{i+1\,j+1}^t) \\
& \vee (a_{i-1\,j-1}^t \wedge a_{i-1\,j}^t \wedge \bar{a}_{i-1\,j+1}^t \wedge a_{i\,j-1}^t \wedge \bar{a}_{i\,j}^t \wedge \bar{a}_{i\,j+1}^t \wedge \bar{a}_{i+1\,j-1}^t \wedge \bar{a}_{i+1\,j}^t \wedge \bar{a}_{i+1\,j+1}^t) \\
& \vee \cdots \vee (a_{i-1\,j-1}^t \wedge a_{i-1\,j}^t \wedge \bar{a}_{i-1\,j+1}^t \wedge \bar{a}_{i\,j-1}^t \wedge a_{i\,j}^t \wedge \bar{a}_{i\,j+1}^t \wedge \bar{a}_{i+1\,j-1}^t \wedge \bar{a}_{i+1\,j}^t \wedge \bar{a}_{i+1\,j+1}^t) \\
& \vee \cdots \vee (a_{i-1\,j-1}^t \wedge a_{i-1\,j}^t \wedge a_{i-1\,j+1}^t \wedge \bar{a}_{i\,j-1}^t \wedge a_{i\,j}^t \wedge \bar{a}_{i\,j+1}^t \wedge \bar{a}_{i+1\,j-1}^t \wedge \bar{a}_{i+1\,j}^t \wedge \bar{a}_{i+1\,j+1}^t) \vee \cdots
\end{aligned}
$$

The first two terms displayed in that formula require $a_{i\,j}^t$ to be false and hence there must be exactly three of the other cells without negations. Therefore there are $\binom{8}{3} = 56$ terms of this type. The last two terms have $a_{i\,j}^t$ true and so there can be either 2 or 3 other cells without the bar; hence there are $\binom{8}{2} + \binom{8}{3} = 28 + 56 = 84$ terms of this type yielding 140 terms total. Using the fuzzy interpretation for the logical functions described above, we can implement Life embedded in a uniformly fuzzy background.

Logic 1: CFMQVS

or	0	0.25	0.5	0.75	1
0	0	0.25	0.5	0.75	1
0.25	0.25	0.5	0.75	1	1
0.5	0.5	0.75	1	1	1
0.75	0.75	1	1	1	1
1	1	1	1	1	1

and	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.25	0	0.0625	0.125	0.1875	0.25
0.5	0	0.125	0.25	0.375	0.5
0.75	0	0.1875	0.375	0.5625	0.75
1	0	0.25	0.5	0.75	1

Logic 2: Max/Min

or	0	0.25	0.5	0.75	1
0	0	0.25	0.5	0.75	1
0.25	0.25	0.25	0.5	0.75	1
0.5	0.5	0.5	0.5	0.75	1
0.75	0.75	0.75	0.75	0.75	1
1	1	1	1	1	1

and	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.25	0	0.25	0.25	0.25	0.25
0.5	0	0.25	0.5	0.5	0.5
0.75	0	0.25	0.5	0.75	0.75
1	0	0.25	0.5	0.75	1

Logic 3: Probabilistic

or	0	0.25	0.5	0.75	1
0	0	0.25	0.5	0.75	1
0.25	0.25	0.4375	0.625	0.8125	1
0.5	0.5	0.625	0.75	0.875	1
0.75	0.75	0.8125	0.875	0.9375	1
1	1	1	1	1	1

and	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.25	0	0.0625	0.125	0.1875	0.25
0.5	0	0.125	0.25	0.375	0.5
0.75	0	0.1875	0.375	0.5625	0.75
1	0	0.25	0.5	0.75	1

Logic 4: MV

or	0	0.25	0.5	0.75	1
0	0	0.25	0.5	0.75	1
0.25	0.25	0.5	0.75	1	1
0.5	0.5	0.75	1	1	1
0.75	0.75	1	1	1	1
1	1	1	1	1	1

and	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.25	0	0	0	0	0.25
0.5	0	0	0	0.25	0.5
0.75	0	0	0.25	0.5	0.75
1	0	0.25	0.5	0.75	1

Logic 5: gcd/lcm

or	0	0.25	0.5	0.75	1
0	0	0.25	0.5	0.75	1
0.25	0.25	0.25	0.25	0.25	0.25
0.5	0.5	0.25	0.5	0.25	0.5
0.75	0.75	0.25	0.25	0.75	0.25
1	1	0.25	0.5	0.25	1

and	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.25	0	0.25	0.5	0.75	1
0.5	0	0.5	0.5	1.5	1
0.75	0	0.75	1.5	0.75	3
1	0	1	1	3	1

Table IV. Some Fuzzy Logic Function Values

When we use our five logics to create various fuzzy versions of the Game of Life, we generally see expected behavior given our experience with one dimensional automata. Logic 1 generates fuzzy automata that tend toward homogeneous fuzzy behavior which in one dimension [5] was consistent with chaotic or complex behavior. It is also true that Logic 2 leads quickly to homogeneous fuzzy behavior. Logic 3 leads to homogeneous fuzzy values near 0. Logic 4 becomes Boolean and the ordinary Game of Life results. Another fuzzy generalization of the Game of Life appears in [17].

The most interesting generalization from our viewpoint is Logic 5. It leads to a growing Boolean region with fuzzy behavior 1/2 outside the region. The growing edge of the Boolean region leads to an alternation between on and off which form waves, in the absence of interference, and ultimately these waves interact with the initial Boolean configuration. In Figure 10 we see this fuzzy automaton iterated upon a 5 by 5 random initial configuration (generated by the J [8] expression `?. 5 5$2`). Notice that at iterations two to four the alive configuration is rather trivial. By iterations 29 and 30 the behavior seems quite complex. There is some appearance of diagonal density and the waves near the edges are clearly visible. However, there is remarkably little symmetry and the configuration seems to be rather far from settling into any pattern. Figure 11 shows the status after 100 iterations and [10] has a link to an animation showing the time evolution of this 2-dimensional automaton.

Much as we did in the one-dimensional situation, we can define the repeated input response function for two dimensional automata. For 3 by 3 neighborhoods we need to make all 9 input values equal. Because of the discontinuous nature of Logic 5, it is unsuitable for constructing a response function. However, the repeated input response function for the Game of Life for two other choices of fuzzy logic is shown in Figure 12. The response function for Logic 1 has an attractive fixed point near 0.37. Fuzzy Logic 3 has what appears to be a fixed point near 0.25 that is tantalizingly close to being a double fixed point. However, the response function falls just short of the identity function and so the only fixed point is at 0. Indeed, we see Logic 1 and Logic 3 have long term behavior on Fuzzy Life that is completely consistent with our expectations given their response functions.

7. Conclusions

Fuzzy automata offer a mechanism for new classifications of automata and the real dynamics of the repeated input response functions for these fuzzy automata can be used for insight into their qualitative behavior. The construction of fuzzy automata can be generalized to different fuzzy logics and there is no problem in extending the construction to two dimensional automata. Rule entropy can also be extended. We have seen that the relationship between rule entropy and the behavior of the fuzzy Game of Life are consistent with expectations developed from the one dimensional case although the response functions seem more closely related to qualitative behavior in general.

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References

1. H. Gutowitz: Cellular Automata and the Sciences of Complexity (Part I), Complexity, 1 5, pp 16-22, 1996.
2. H-O. Peitgen, H. Jürgens and D. Saupe: Chaos and Fractals. Springer-Verlag, New York, 1992.
3. S. Wolfram: Cellular Automata and Complexity, collected papers. Addison-Wesley Publishing Company, Reading, 1994.
4. S. Wolfram: Universality and Complexity in Cellular Automata. Physica D, 10, pp 1-35

1984.
5. G. Cattaneo, P. Flocchini, G. Mauri, C. Quaranta Vogliotti, and N. Santoro: Cellular automata in fuzzy backgrounds. Physica D, 105, pp 105-120, 1997.
6. W. K. Wooters and C. G. Langton: Is there a sharp phase transition for deterministic automata?. Physica D, 45, pp 95-104 1990.
7. P. Grim, G. Mar, and P. St. Denis: The Philosophical Computer. MIT Press, Cambridge, Massachusetts, 1998.
8. K. E. Iverson: J Introduction and Dictionary, http://www.jsoftware.com. Iverson Software Inc., Toronto, 1995.
9. C. Reiter: Fractals, Visualization and J, 2nd edition. Jsoftware, Inc, Toronto, 2000.
10. C. Reiter: Fuzzy Automata, http://www.lafayette.edu/~reiterc/mvp/fuzz_auto/index.html.
11. D. Dubois and H. Prade: Fuzzy Sets and Systems. Academic Press, New York, 1980.
12. G. J. Klir, U. H. St. Clair, and B. Yuan: Fuzzy Set Theory. Prentice Hall, 1997.
13. C. A. Reiter: Sierpinski Fractals and GCDs. Computers & Graphics, 18 6, pp 885-891 1994.
14. E. Berlekamp, J. Conway, and R. Guy: Winning Ways For Your Mathematical Plays. Academic Press, New York, 1982.
15. M. Gardner: The fantastic combinations of John Conway's new solitaire game of "life". Scientific American, 223 4, pp 120-123, 1970.
16. M. Gardner: On cellular automata, self-replication, the Garden of Eden and the game "life". Scientific American, 224 4, pp 112-117, 1971.
17. G. Mar, and P. St. Denis: Real Life. International Journal of Bifurcation and Chaos, 6 11, pp 2077-2086, 1996.
18. S. Wolfram: Computation Theory of Cellular Automata. Communications in Mathematical Physics, 96, pp 15-57, 1984.

Appendix: Rule Entropy

Rule entropy was defined in [5] and will be explicitly given below. Informally, rule entropy can be thought of as a measure based upon the possible outputs of the initial configurations and as such it is comparable to the lambda parameter of [6]. However, it uses some of the distribution information in one time step rather than just total frequency. Thus, rule entropy would seem to have somewhat more information than the lambda parameter but far less than classical entropies that would measure the distribution averaged over many time steps after transient behavior is removed. While "rule entropy" may not be the best name for this measure of fuzziness, we follow [5] and use that terminology. Since there are connections between the lambda parameter and entropy, it is not surprising that rule entropy, which is intermediate, is also related to these measures. We consider rule entropy in one dimension, compute it for some of our examples and then generalize it to two dimensions in a way that allows us to compute the rule entropy for the Game of Life.

For one dimensional automata, rule entropy is defined as a sum of left and right entropies; each one sided entropy is in turn the sum of level 1 and 2 one-sided entropies that arise by varying two or one of the cells on the other side. For convenience we take $p^l_{\vec{i}}$ to be the proportion of states with $\vec{i}$ as a suffix that result in a one. Then

$$H^l_1 = -\sum_{<a,b>} p^l_{<a,b>} \log_2(p^l_{<a,b>}) + (1-p^l_{<a,b>})\log_2(1-p^l_{<a,b>})$$

where *a* and *b* can be 0 or 1. We can use the mnemonic that the superscript indicates which side to vary and the subscript the number of coordinates to vary. Similarly,

$$H^l_2 = -\sum_{<a>} p^l_{<a>} \log_2(p^l_{<a>}) + (1-p^l_{<a>})\log_2(1-p^l_{<a>})\ .$$

Then the left rule entropy is $H^l = H^l_1 + H^l_2$. The right entropy is defined in an analogous manner:

we take $p^r_{\vec{i}}$ to be the proportion of states with $\vec{i}$ as a prefix that result in a one. Then

$$H_1^r = -\sum_{<a,b>} p^r_{<a,b>} \log_2(p^r_{<a,b>}) + (1-p^r_{<a,b>})\log_2(1-p^r_{<a,b>})$$

and

$$H_2^r = -\sum_{<a>} p^r_{<a>} \log_2(p^r_{<a>}) + (1-p^r_{<a>})\log_2(1-p^r_{<a>})$$

and the right rule entropy is $H^r = H_1^r + H_2^r$. As an illustration we will compute the rule entropy for Rule 17. The vector index $<0,0>$ is a suffix to both the triples resulting in one, hence $p^l_{<0,0>} = 2/2 = 1$. Likewise, we see $p^l_{<0,1>} = 0/2$, $p^l_{<1,0>} = 0/2$, and $p^l_{<1,1>} = 0/2$ resulting in $H_1^l = 0$ since for each term either the proportion or its logarithm is 0. Also, $p^l_{<0>} = 2/4$ since both true outputs have $<0>$ as a suffix. We can check $p^l_{<1>} = 0/4$ and since $\frac{1}{2}\log_2(\frac{1}{2}) = -\frac{1}{2}$ we see $H_2^l = 1$. The analogous computations on the right are $p^r_{<0,0>} = 1/2$, $p^r_{<0,1>} = 0/2$, $p^r_{<1,0>} = 1/2$, $p^r_{<1,1>} = 0/2$ so $H_1^l = 2$. Also, $p^r_{<0>} = 1/4$, $p^r_{<1>} = 1/4$ so $H_2^r = -2(\frac{1}{4}\log_2(\frac{1}{4}) + \frac{3}{4}\log_2(\frac{3}{4})) \approx 1.62$ and hence $H^r \approx 3.62$. The table of right and left rule entropies appearing in the paper [5] appear to skip the $\frac{3}{4}\log_2(\frac{3}{4})$ terms. However, this does not substantially impact their conclusions.

rule	H^l	H^r
17	1	3.62
73	4.8	4.8
125	3	3.6
146	4.8	4.8

Table V. Selected rule entropies

Table V summarizes the rule entropies for the examples of fuzzy automata that we saw in Figure 1. Rule entropies near the maximal value of 6 are expected [5] to be associated with homogeneous fuzzy behavior, zero rule entropy with expanding Boolean windows and intermediate values associated with more complex interaction. Here we see Rule 17 has the low rule entropy and the Boolean behavior dominates. However, Rules 73, 125 and 146 all have relatively intermediate rule entropies and fairly complex behavior. It is somewhat disappointing that Rule 125 has lower rule entropy given the expectation that homogeneous behavior corresponds to rule entropy near the maximal value 6. This illustrates that it isn't possible to offer a simple threshold on rule entropy to distinguish these behaviors. Indeed, we saw that the qualitative behaviors can vary quite a lot when the fuzzy logic is changed. Rule entropy can be viewed as a type of entropy that does not account for the patterns excluded by the repeated iteration of the rule [3, 18] and in that sense is naive. Nonetheless, it is extremely easy to compute and reasonably effective given those qualifications.

We can define rule entropy for automata in two dimensions in various ways that seem compatible with the choice in one dimension. For Boolean automata defined on 3 by 3 neighborhoods centered on the cell being updated, it seems appropriate to maintain the directional distinctions offered in one dimension. Thus, for a 3 by *i* matrix *I*, where *i* is 1 or 2, we define p^r_I to be the proportion of 3 by 3 input configurations with *I* appearing on the left and which result in life. Thus, we basically consider proportions arising from fixing two or one columns on the left. Then

$$H_1^r = -\sum_{I \text{ is 3 by 2}} p_I^r \log_2(p_I^r) + (1-p_I^r)\log_2(1-p_I^r),$$

and

$$H_2^r = -\sum_{I \text{ is 3 by 1}} p_I^r \log_2(p_I^r) + (1-p_I^r)\log_2(1-p_I^r)$$

with $H^r = H_1^r + H_2^r$ as before. Left, top and bottom entropies can be defined in an analogous manner. However, by symmetry, these will all be the same for the Game of Life. In order to compute the right rule entropy for the Game of Life, we need to consider 2^6 fixed 3 by 2 arrays for H_1^r and 2^3 fixed arrays for H_2^r. We can organize our computation as follows: consider the 3 by 2 arrays keeping track of whether the center is lit and how many of the other 5 fixed positions are lit. Then count the number (out of the eight ways) that the three by three neighborhood can be completed to obtain life.

In Table VI, *C* denotes the value of the center, *N* denotes the number of ones in the other 5 positions being fixed and *U* denotes the number of ways those ones can be rearranged in those 5 positions. For example, when *C* = 1, *N* = 1, we need one or two cells lit in the last column (the three varied positions). Since $\binom{3}{1}+\binom{3}{2}=6$ we have $p_I^r = 6/8$. Because there are five ways to rearrange the fixed noncentral 1, we will obtain five terms with this proportion.

In Table VII, we summarize the computations for level 2 rule entropy. In that table, *N* denotes the number of the fixed three positions that contain a 1, *U* denotes the number of ways to rearrange those ones, *C*0 designates the number of ways to fill in the varied 6 positions to result in life if the center must be 0, *C*1 designates the number of ways to fill in the varied 6 positions to result in life if the center must be 1.

This results in a rule entropy of $H^r \approx 46.87$. The maximal possible rule entropy for Boolean 3 by 3 rule is 72 since it is possible to create an automata with all $2^3 + 2^6 = 72$ values of *I* resulting in $p_I^r = 1/2$. Thus the normalized rule entropy for the Game of Life is approximately 0.65 which is intermediate. The fact that it represents complex behavior that tends to settle down after many iterations is consistent with our remarks above.

C	N	U	p_I^r
0	0	1	$1/8$
1	0	1	$4/8$
0	1	5	$3/8$
1	1	5	$6/8$
0	2	10	$3/8$
1	2	10	$4/8$
0	3	10	$1/8$
1	3	10	$1/8$

Table VI. Computing level 1 right rule entropy for Life.

N	U	$C0$	$C1$	p_I^r
0	1	10	20	$30/64$
1	3	10	15	$25/64$
2	3	5	6	$11/64$
3	1	1	1	$2/64$

Table VII. Computing level 2 right rule entropy for Life.

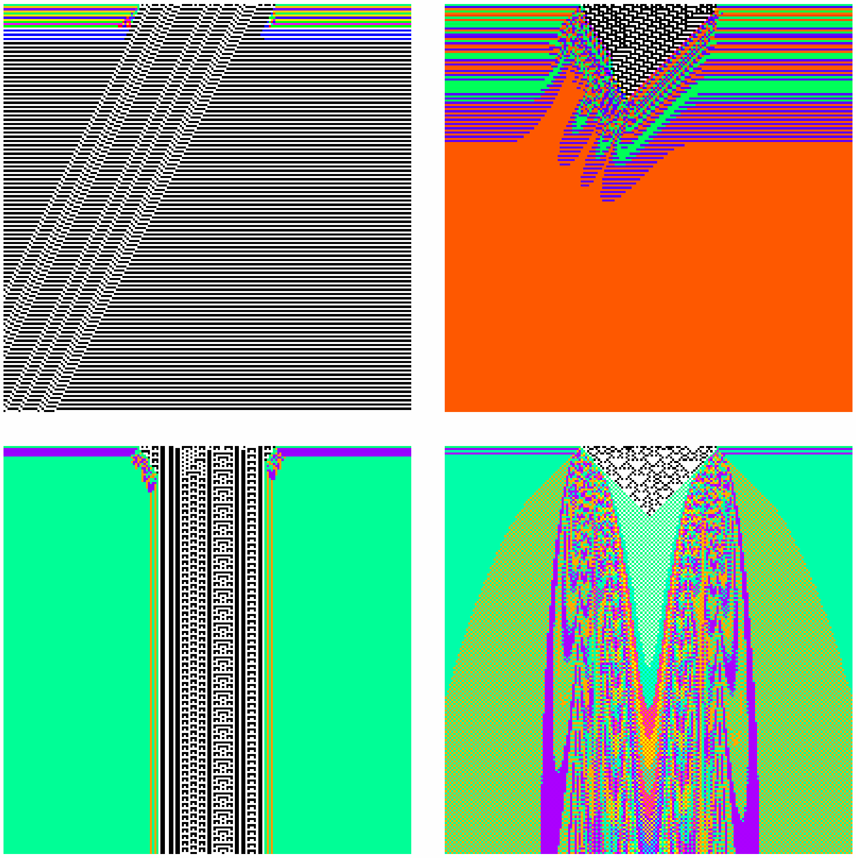

Figure 1. Rules 17, 125, 73, 146 as fuzzy automata.

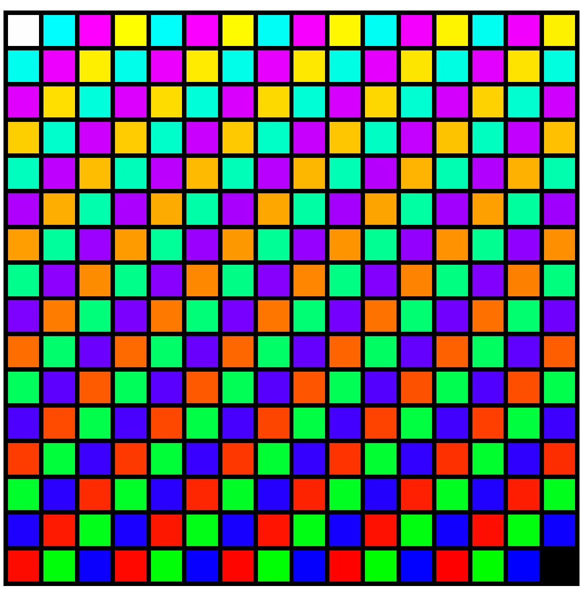

Figure 2. Fuzzy automata palette.

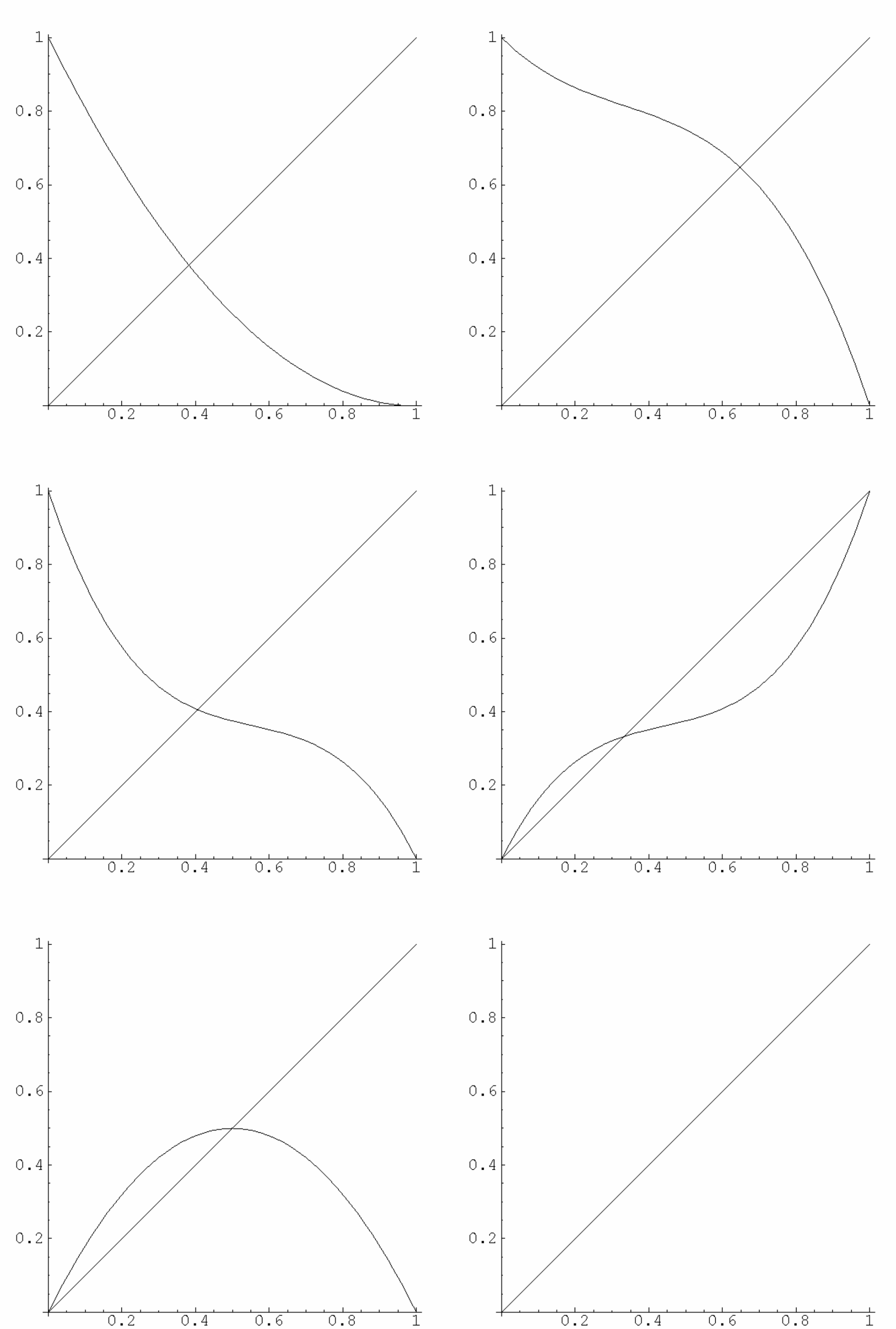

Figure 3. Repeated input response functions for fuzzy rules 17, 125, 73, 146, 90 and 170.

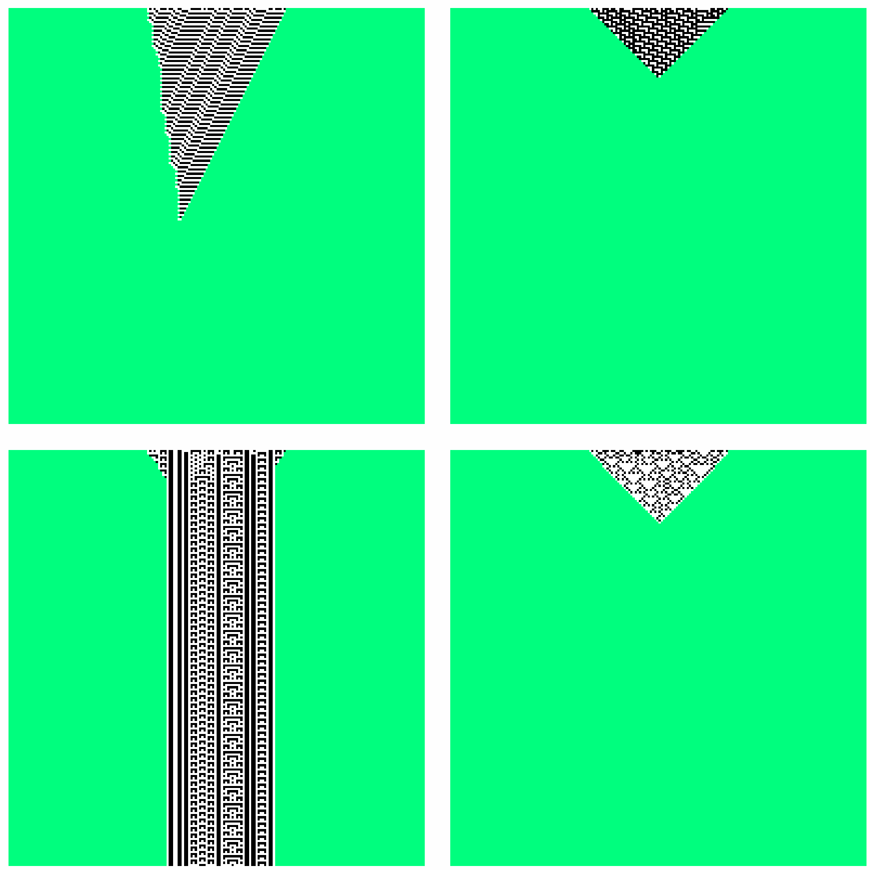

Figure 4. Rules 17, 125, 73 and 146 using Fuzzy Logic 2.

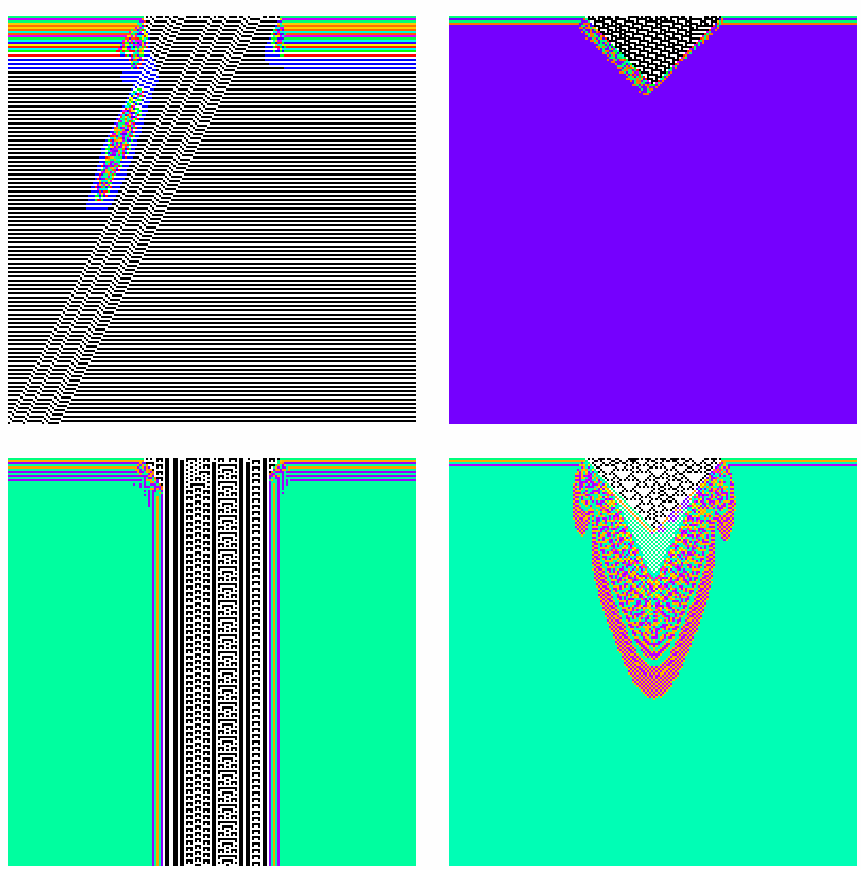

Figure 5. Rules 17, 125, 73 and 146 using Fuzzy Logic 3.

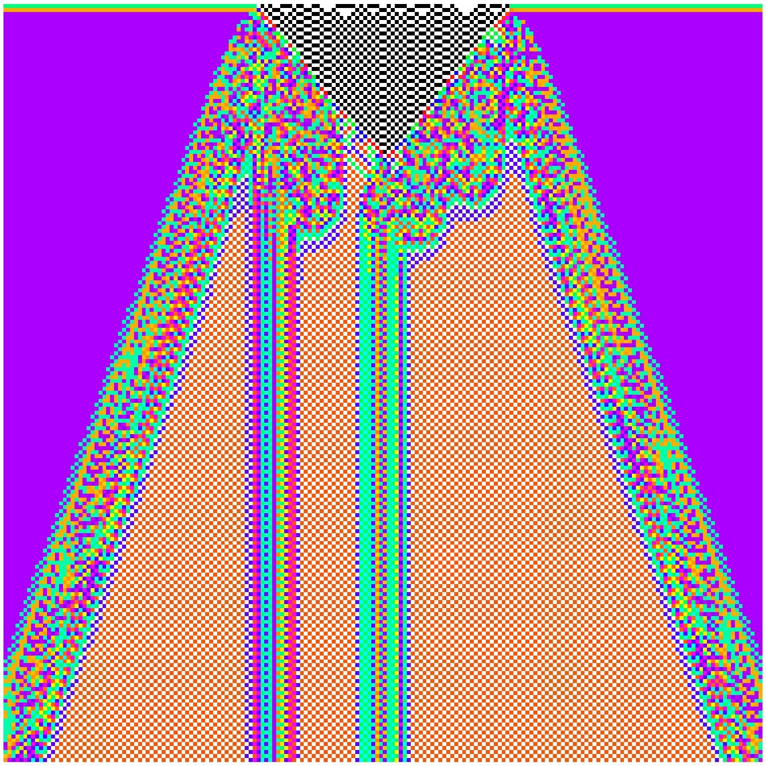

Figure 6. Rule 50 using fuzzy Logic 3.

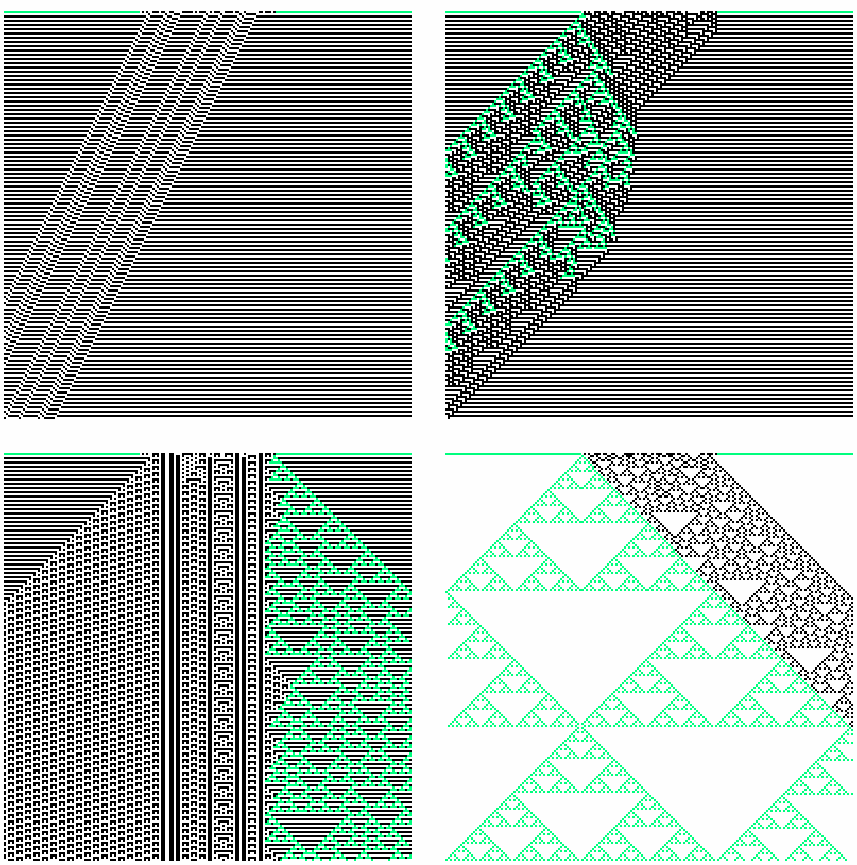

Figure 7. Rules 17, 125, 73 and 146 using Fuzzy Logic 4.

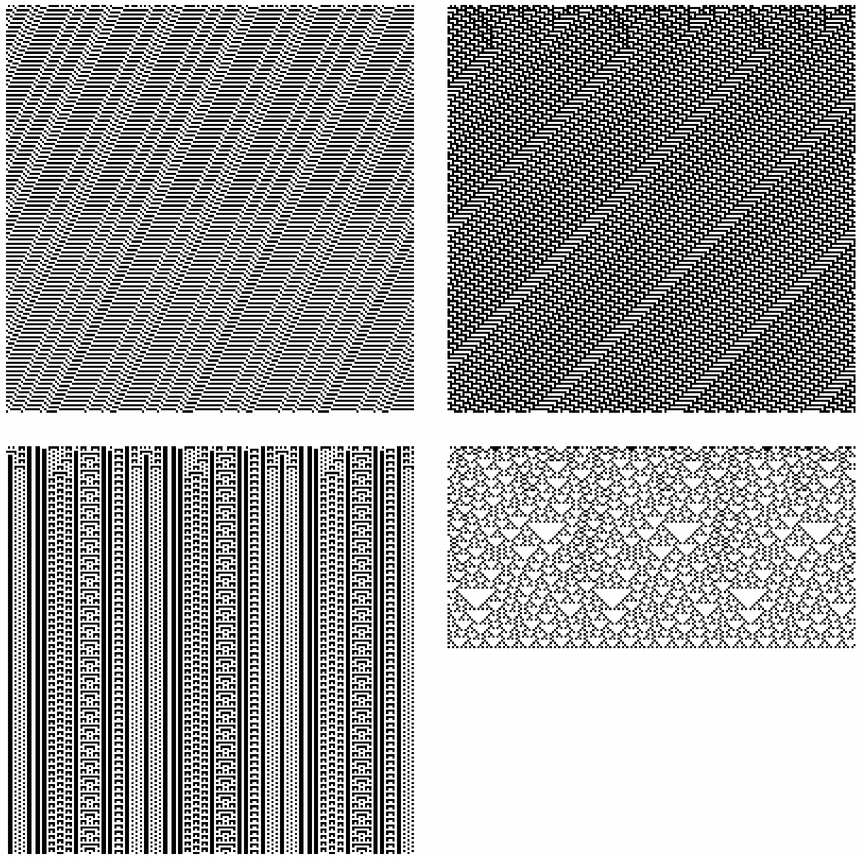

Figure 8. Rules 17, 125, 73 and 146 using Boolean Automata and periodic boundary conditions. Three periods are shown.

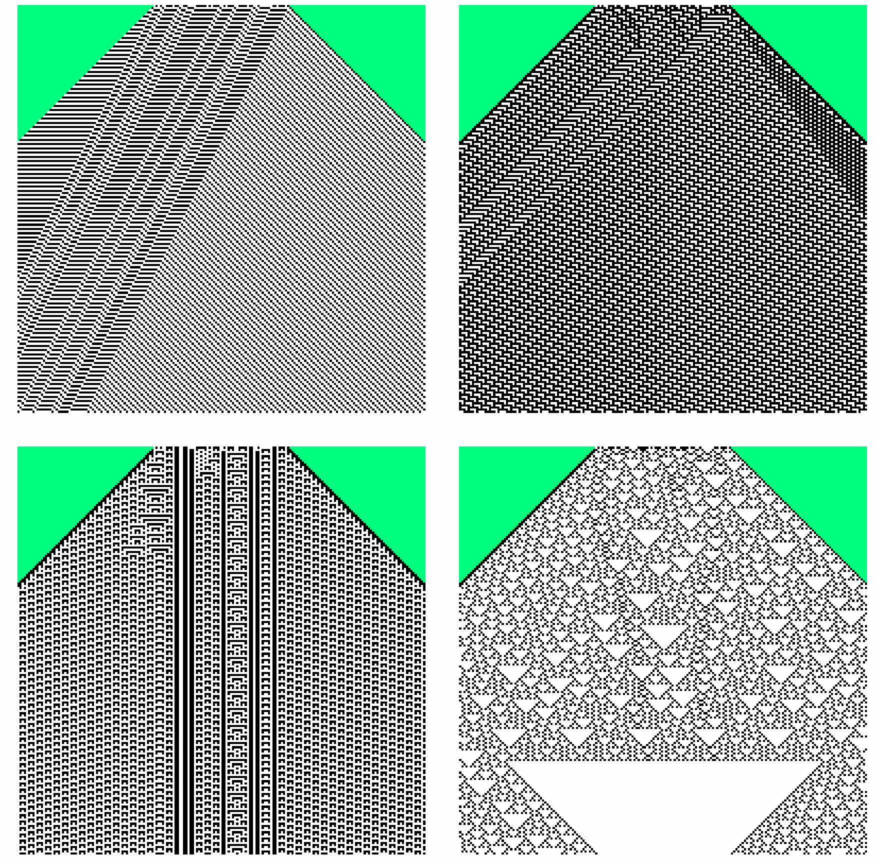

Figure 9. Rules 17, 125, 73 and 146 using Fuzzy Logic 5.

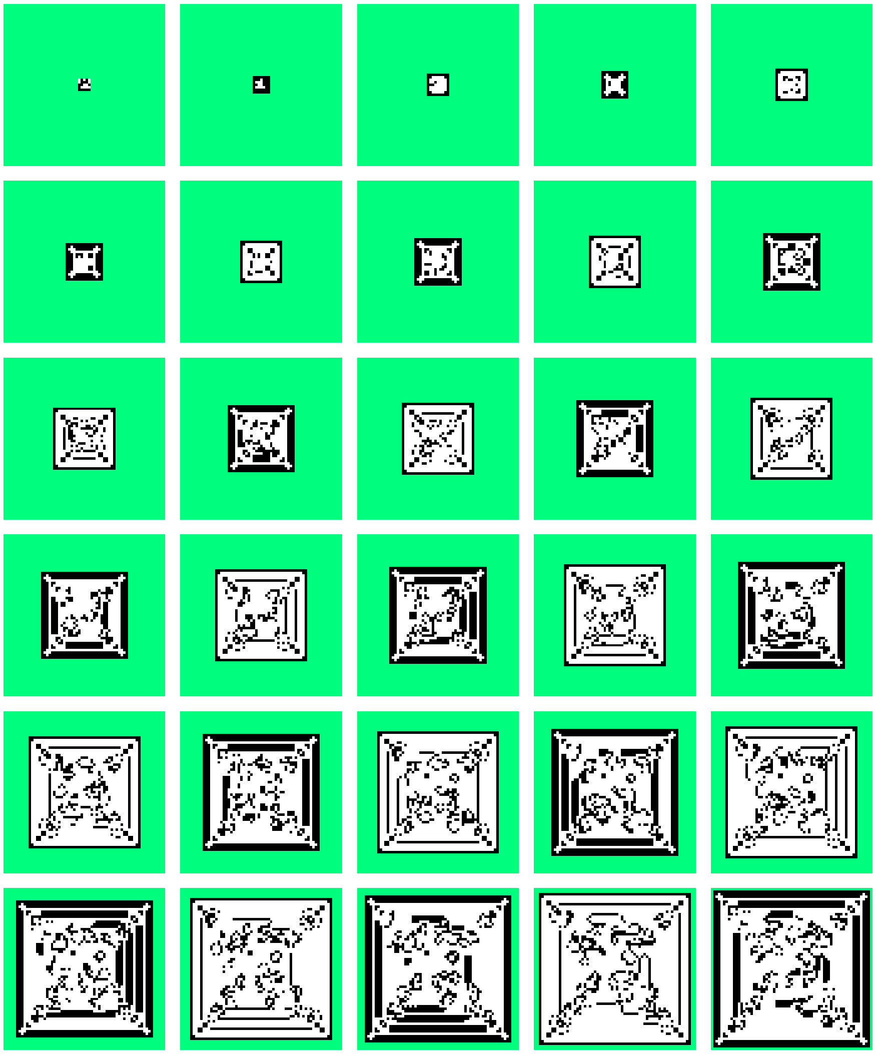

Figure 10. Nurturing the Game of Life using Fuzzy Logic 5.

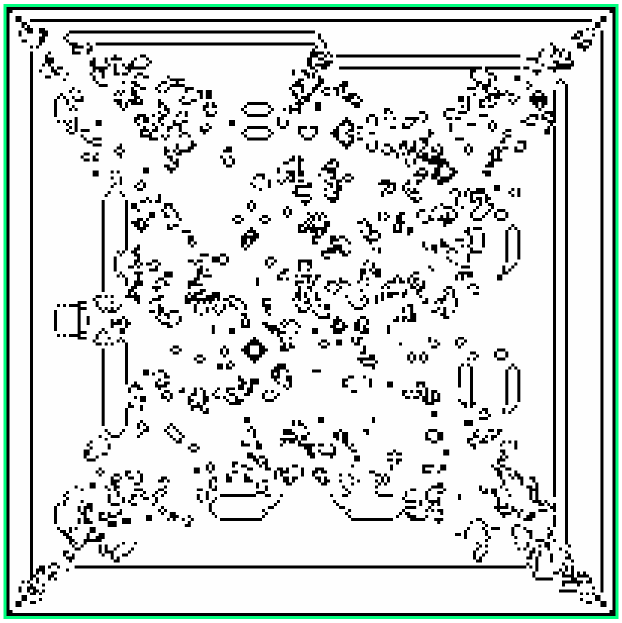

Figure 11. The Fuzzy Game of Life after 100 iterations

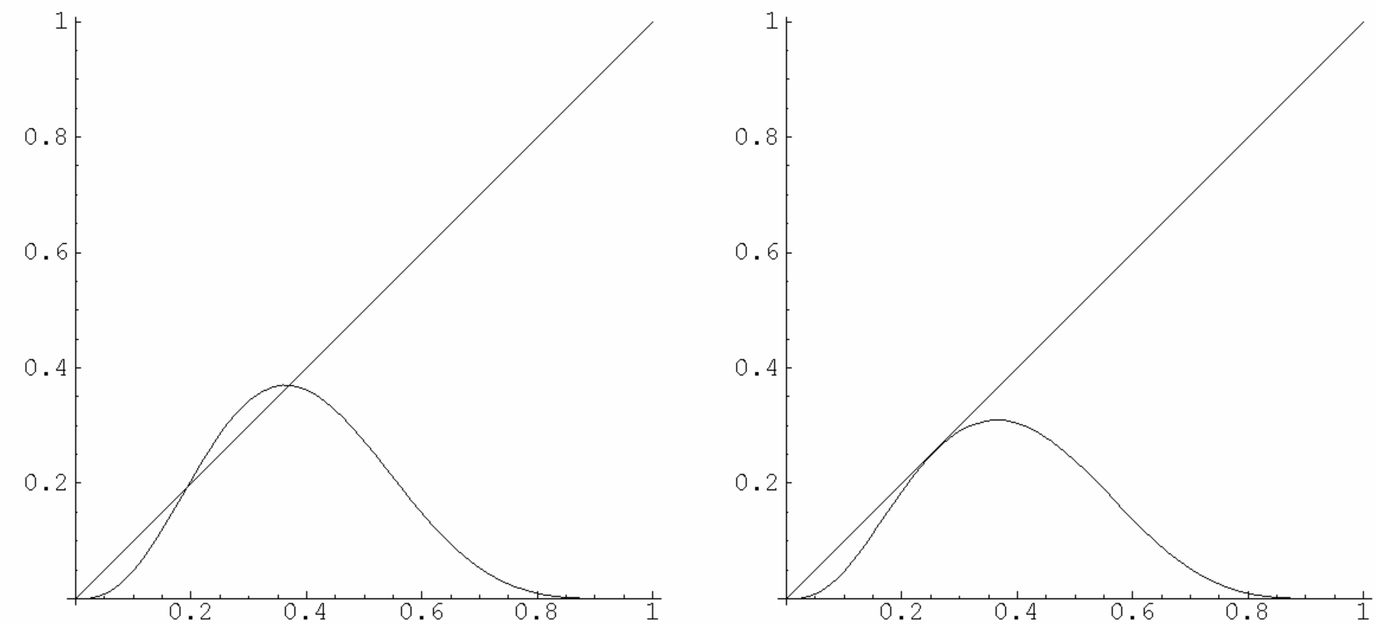

Figure 12. Repeated input response function for the Game of Life using Logics 1 and 3.