FAMILIES OF NEARLY PERFECT PARALLELEPIPEDS (PREPRINT)

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Abstract

It is unknown whether there are perfect parallelepipeds, that is, parallelepipeds with integer-length edges, face diagonals and body diagonals. A stronger version of the problem also requires the coordinates to be integer. In that case, the vectors are integer-length integer vectors. We will show how to extend integer-length integer vectors in any dimension to one higher dimension and utilize that construction to present three parametric families of parallelepipeds that are nearly perfect in the sense that only two conditions need be satisfied in order for the parallelepiped to be perfect. Computer searches show many examples where either, but not both, of those conditions may be satisfied.

1. Introduction

A cuboid is a rectangle box and a cuboid is perfect if its edges, face diagonals and body diagonal are all integer length. It is unknown as to whether there is a perfect cuboid, although many examples are known that satisfy all but one of those conditions [2]. A more general question, also open, asks whether there is a nondegenerate parallelepiped with integer-length edges, face diagonals, and body diagonals [2]. In terms of vectors, if we call the edges $\vec{u}$, $\vec{v}$, $\vec{w}$ then the corresponding 3-dimensional parallelepiped is perfect if those vectors are independent, and all thirteen of the following lengths are integers: $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{w}\|$, $\|\vec{u}+\vec{v}\|$,$\|\vec{u}-\vec{v}\|$, $\|\vec{u}+\vec{w}\|$, $\|\vec{u}-\vec{w}\|$, $\|\vec{v}+\vec{w}\|$, $\|\vec{v}-\vec{w}\|$, $\|\vec{u}+\vec{v}+\vec{w}\|$, $\|-\vec{u}+\vec{v}+\vec{w}\|$, $\|\vec{u}-\vec{v}+\vec{w}\|$, and $\|\vec{u}+\vec{v}-\vec{w}\|$. We will primarily be interested in the stronger version where we also require the coordinates of the vectors to be integer.

In this note we describe a method for extending an integer-length integer vector in one dimension to a parametric family of infinitely many in the next. In the case that a 1-dimensional vector is extended to 2-dimensions, we obtain the familiar parameterization of Pythagorean pairs. We will see that when this extension is done to three dimensions, we obtain a 3-parameter family of cuboids that is nearly perfect in the sense that only two conditions must hold. We also use this idea to give a 3-parameter family of nearly perfect parallelepipeds and lastly generalize the 2-parameter family of nearly perfect parallelepipeds from [3] to a 4-parameter family. In each case we obtain a family of parallelepipeds that

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is nearly perfect in the sense that two expressions in terms of the parameters would need to be squares in order for the parallelepiped to be perfect.

2. Extending Integer Length Integer Vectors and Near Cubiods

First we comment that we will use a comma to indicate that a new coordinate is being adjoined to the end of a vector. Thus, we would interpret $<< 3, 4 >, 12 >$ as $< 3, 4, 12 >$.

Theorem 1. *If $\vec{u}$ is an integer-length integer vector and $k \in \mathbf{Z}$, then $< 2k\vec{u}, \|\vec{u}\|^2 - k^2 >$ is also an integer-length integer vector. In particular, its length is $\|\vec{u}\|^2 + k^2$.*

Proof. The fact that $\vec{u}$ is integer length is equivalent to the fact that $\|\vec{u}\|^2$ is a perfect square. We see

$$\| < 2k\vec{u}, \|\vec{u}\|^2 - k^2 > \|^2 =< 2k\vec{u}, \|\vec{u}\|^2 - k^2 > \cdot < 2k\vec{u}, \|\vec{u}\|^2 - k^2 >$$

$$= 4k^2\|\vec{u}\|^2 + (\|\vec{u}\|^2 - k^2)^2 = (\|\vec{u}\|^2 + k^2)^2$$

which is the desired perfect square and the result follows. □

This theorem allows us to easily generate integer-length integer vectors in any dimension such that any leading portion of the vector is also an integer-length integer vector.

Corollary 2. *If a and b are integers, then $< 2ab, a^2 - b^2 >$ is an integer-length integer vector. That is, it is a Pythagorean pair.*

Proof. The 1-dimensional vector $< a >$ has length $|a|$ and hence is an integer length-integer vector. Applying Theorem 1 with b in place of the parameter k gives the integer vector $< 2ab, a^2 - b^2 >$ that has integer length $a^2 + b^2$. □

This gives the usual parameterization of Pythagorean triples, without the usual parity and primitivity conditions. A vector (with more than one coordinate) is primitive if there is no common divisor of its coordinates larger than 1. For our work with parallelepipeds, we will need negative entries and expect some of the edge vectors may not be primitive.

Corollary 3. *If a, b, and c are integers, then $< 4abc, 2c(a^2 - b^2), (a^2 + b^2)^2 - c^2 >$ is an integer-length integer vector of length $(a^2 + b^2)^2 + c^2$.*

Proof. The result follows from Theorem 1 using $< 2ab, a^2 - b^2 >$ as the integer length integer vector (as per Corollary 2) and with parameter c. □

Corollary 4. *Let a, b, and c be integers such that x, y, and z, defined as follows, are nonzero.*

$$x = 4abc, \;\; y = 2c(a^2 - b^2), \;\; z = (a^2 + b^2)^2 - c^2$$

Then $|x|$, $|y|$, and $|z|$ give the sides of a perfect cuboid iff $x^2 + z^2$ and $y^2 + z^2$ are perfect squares.

Proof. In light of Corollaries 2 and 3, $< x, y >$, $< x, y, z >$ are integer-length integer vectors and hence we need only check that $< x, z >$ and $< y, z >$ are integer-length integer vectors. □

While parametric families that satisfy all but one condition are known [2], Corollary 4 shows how the construction in Theorem 1 can be used in a routine way to obtain a family of cuboids that satisfies all except two conditions. In the next section we will see similar results for two families of parallelepipeds.

Also, while extensive searches for perfect cuboids have been done by others, we also searched all cases of $1 \leq a, b, c \leq 1440$ finding 1090 nondegenerate cuboids satisfying an additional, but not both, conditions. There where 574 cases where $< x, z >$ was also an integer length integer vector and 516 cases where $< y, z >$ was, nearly an even split.

The first example found by the search was $< x, y, z >=< 448, 840, 495 >$ which satisfies all the conditions except $x^2 + z^2 = 445729$, which is not a square.

3. Families of Nearly Perfect Parallelepipeds

Proposition 5. *Let a, b, c, x, y and z be as described in Corollary 4, then let*

$$\vec{u} =< x, y, 0 >, \ \vec{v} =< x, -y, 0 >, \ \vec{w} =< 0, 0, z > .$$

Then $\vec{u}$, $\vec{v}$, $\vec{w}$ form a nondegenerate perfect parallelepiped iff $\|\vec{u} + \vec{v} \pm \vec{w}\|^2 = 4x^2 + z^2$ and $\|\vec{u} - \vec{v} \pm \vec{w}\|^2 = 4y^2 + z^2$ are perfect squares.

Proof. First notice these vectors are independent so long as x, y and z are nonzero, as per the description. The construction of x, y and z also assures that $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{u} \pm \vec{w}\|$, $\|\vec{v} \pm \vec{w}\|$ are all integer. The lengths of $\|\vec{w}\|$, and $\|\vec{u} \pm \vec{v}\|$ are integer because they only have one nonzero coordinate. It remains to check the four body diagonals where it is clear the sign of $\vec{w}$ in the sum does not change the length, thus there are two conditions to check, which are as noted above. □

We also searched all cases of the construction in Proposition 5 for $1 \leq a, b, c \leq 1440$ finding 1358 nondegenerate parallelepipeds satisfying an additional, but not both, conditions. There where 644 cases where $4x^2 + z^2$ was a square and 714 cases where $4y^2 + z^2$ was.

The final family of near parallelepipeds that we construct has a different form. It is based upon repeated applications of Corollary 2 blended in a certain way, rather than Corollary 3. Notice that in Corollary 2, two parameters generate two coordinates (a Pythagorean pair) and hence could be viewed as a map from $\mathbf{Z}^2$ into $\mathbf{Z}^2$. The next proposition essentially takes two pairs of integers as input and produces two pairs, one pair by iterating Corollary 2 twice (the second time is done explicitly since its length is used too) and another by applying it once. Then these are combined to form a four parameter family of parallelepipeds that would be perfect if two additional constraints held. This family generalizes the two parameter family that was given in Proposition 6 in [3].

Proposition 6. *Let a, b, c, and d be integers chosen so that none of the following are zero: $p = 2ab$, $q = a^2 - b^2$, $r = 2cd$, $s = c^2 - d^2$. Then let*

$$\vec{u} = (r^2 + s^2) < p, q, 0 >, \ \vec{v} = (r^2 + s^2) < -p, q, 0 >, \ \vec{w} = 2p < r^2 - s^2, 0, 2rs > .$$

Then $\vec{u}$, $\vec{v}$, $\vec{w}$ form a nondegenerate perfect parallelepiped iff $\|\vec{u} + \vec{w}\| = \|\vec{v} - \vec{w}\|$ and $\|\vec{u} - \vec{w}\| = \|\vec{v} + \vec{w}\|$ are perfect squares.

Proof. It is easy to see $\vec{u}$, $\vec{v}$, $\vec{w}$ are independent vectors so long as all the coordinates are nonzero. The construction of p and q assures that $< p, q, 0 >$ is integer length. Thus, we readily see that $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{w}\|$ and $\|\vec{u} \pm \vec{w}\|$ are all integer. It is a bit more work to check the lengths of the body diagonals, but these are integers:

$$\|\vec{u} + \vec{v} \pm \vec{w}\| = 2(a^2 + b^2)(c^2 + d^2)^2$$

$$\|\vec{u} - \vec{v} + \vec{w}\| = 8ab(c - d)(c + d)(c^2 + d^2)$$

$$\|\vec{u} - \vec{v} - \vec{w}\| = 16abcd(c^2 + d^2)$$

A *Mathematica* notebook verifying those facts may be found at [1]. Notice none of those lengths are zero due to the fact that p, q, r and s were assumed to be nonzero. Also, $\|\vec{u}+\vec{w}\| = \|\vec{v}-\vec{w}\|$ and $\|\vec{u}-\vec{w}\| = \|\vec{v}+\vec{w}\|$ may be verified by direct computation and seen at [1]. Thus, only two conditions remain to obtain a perfect parallelepiped as required. □

We searched all cases of the construction in Proposition 6 for $1 \leq a, b, c, d \leq 256$ finding 19,892 nondegenerate parallelepipeds satisfying an additional, but not both, conditions. Again the conditions were satisfied in roughly equal amounts: 10,524 and 9,368 respectively.

The first example found was $\vec{u} =< 92480, 133518, 0 >$, $\vec{v} =< 92480, -133518, 0 >$ and $\vec{w} =< 0, 148764, -221760 >$ which satisfies all the conditions except $\|\vec{u}+\vec{w}\|^2 = \|\vec{v}-\vec{w}\|^2 =$ 137,413,175,524 which is not a square.

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